Week 9 Exercises

- (Boothby pg. 205 #1) For V a vector space with $\dim V > 1$, show $\mathcal{T}^r(V) = \bigwedge^r(V) \oplus \sum^r(V)$ when $r = 2$ but not for $r > 2$.

Solution: Let $\mathcal{S}$ and $\mathcal{A}$ be the symmetrization and antisymmetrization operators. When $r \geq 2$, $\mathcal{S}$ vanishes on $\bigwedge^r(V)$ and $\mathcal{A}$ vanishes on $\sum^r(V)$. If $\phi \in \mathcal{T}^r(V)$ is the sum of a symmetric α and antisymmetric β, then α, β are uniquely given by $\mathcal{S}(\phi) = \mathcal{S}(\alpha) + \mathcal{S}(\beta) = \alpha$ and $\mathcal{A}(\phi) = \mathcal{A}(\alpha) + \mathcal{A}(\beta) = \beta$. When $r = 2$:

$$\mathcal{S}(\phi)(v,w) + \mathcal{A}(\phi)(v,w) = \frac{1}{2}\left(\phi(v,w) + \phi(w,v)\right) + \frac{1}{2}\left(\phi(v,w) - \phi(w,v)\right) = \phi(v,w)$$

which shows $\mathcal{T}^2(V) = \bigwedge^2(V) \oplus \sum^2(V)$

Assuming $\dim V = n > 1$, there are $\omega, \nu \in V^*$ which are linearly independent. Put

$$\phi = \nu \otimes \omega \otimes \omega \otimes \cdots \otimes \omega.$$

Then $\mathcal{A}(\phi) = 0$ but $\mathcal{S}(\phi) \neq \phi$ since ϕ is not symmetric. This ϕ is not a sum of a symmetric and asymmetric tensor.

Another way to see $\mathcal{T}^r(V) \neq \bigwedge^r(V) \oplus \sum^r(V)$ when $r > 2$ is to count dimensions: First, $\dim \mathcal{T}^r(V) = n^r$. We showed $\dim \bigwedge^r(V) = \binom{n}{r}$. A good combinatorics exercise is to show $\dim \sum^r(V) = \binom{n+r-1}{r}$, and then show that $\binom{n}{r} + \binom{n+r-1}{r} < n^r$ when $r > 2$ and $n > 1$.

- (Boothby pg. 214 #6) Definition and basic properties of interior product. The hard part is the product rule. One way to prove this is 'by force', which Lee does in Proposition 8.13. You should prove it in the special cases when $r = 1$ and $s = 1, 2, 3$. Then, either prove (or accept) that it's true for $r = 1$ and all $s > 1$ - it should be clear at this point, and only the notation makes it hard. Then you can use induction on r, proving that if the product rule holds for r and all s, then it holds for $r + 1$ and all s.

  Extending to tensor fields on a manifold is trivial. Just replace v's with X's. It's not worth re-writing everything.

Solution: First, see $\iota(v)\varphi \in \bigwedge^{r-1}(V)$.

$$\begin{aligned}(\iota(v)\varphi)(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_{r-1}) &= \varphi(v, v_1, \ldots, v_i, \ldots, v_j, \ldots, v_{r-1}) \\ &= -\varphi(v, v_1, \ldots, v_j, \ldots, v_i, \ldots, v_{r-1}) = -(\iota(v)\varphi)(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_{r-1})\end{aligned}$$

Next, show $\iota(v)$ is linear:

$$\begin{aligned}\iota(v)(\varphi + c\psi)(v_1, \ldots, v_{r-1}) &= (\varphi + c\psi)(v, v_1, \ldots, v_{r-1}) = \\ &\varphi(v, v_1, \ldots, v_{r-1}) + c\psi(v, v_1, \ldots, v_{r-1}) = (\iota(v)\varphi + c\iota(v)\psi)(v_1, \ldots, v_{r-1})\end{aligned}$$

Finally, the product rule. Note that for $\varphi \in \bigwedge^1(V)$, $\iota(v)\varphi$ is the scalar $\varphi(v)$.

When $r = 1$, we have:

$$\begin{aligned}
\iota(v)(\varphi \wedge \psi)(w_1, \dots, w_s) &= (\varphi \wedge \psi)(v, w_1, \dots, w_s) \\
&= \varphi(v)\psi(w_1, \dots, w_s) - \sum_{i=1}^{s} \varphi(w_i)\psi(w_1, \dots \overset{i^{th}\text{ spot}}{v} \dots, w_s) \\
&= \varphi(v)\psi(w_1, \dots, w_s) + \sum_{i=1}^{s} (-1)^i \varphi(w_i)\psi(v, w_1, \dots \hat{w}_i \dots, w_s) \\
&= \varphi(v)\psi(w_1, \dots, w_s) + \sum_{i=1}^{s} (-1)^i \varphi(w_i)\iota(v)\psi(w_1, \dots \hat{w}_i \dots, w_s) \\
&= (\iota(v)\varphi \wedge \psi - \varphi \wedge \iota(v)\psi)(w_1, \dots, w_s)
\end{aligned}$$

where we divide up the permutations of $(v, w_1, \dots, w_s)$ into two types, those which fix v and those which first exchange v with w_i.

Now we prove the general product rule by induction on r. The idea is to change a wedge of $r + 1$ and s forms into a wedge of r and $s + 1$ forms.

Suppose the product rule holds for a particular r and for all s. The case $r = 1$ is done above. Let $\phi \in \bigwedge^{r+1}(V)$. Then ϕ is a sum of terms of the form $\theta \wedge \rho$, where θ is a 1-form and ρ is an r form. By linearity, we only need to prove the product rule for one such term. For $\psi \in \bigwedge^s(V)$, we have:

$$\begin{aligned}
\iota(v)(\phi \wedge \psi) &= \iota(v)((\theta \wedge \rho) \wedge \psi) = \iota(v)(\theta \wedge (\rho \wedge \psi)) \\
&= \iota(v)\theta \wedge (\rho \wedge \psi) - \theta \wedge \iota(v)(\rho \wedge \psi) \\
&= \iota(v)\theta \wedge (\rho \wedge \psi) - \theta \wedge [\iota(v)\rho \wedge \psi + (-1)^r \rho \wedge \iota(v)\psi)] \\
&= [\iota(v)\theta \wedge \rho - \theta \wedge \iota(v)\rho] \wedge \psi + (-1)^{r+1}\theta \wedge \rho \wedge \iota(v)\psi \\
&= \iota(v)\phi \wedge \psi + (-1)^{r+1}\phi \wedge \iota(v)\psi
\end{aligned}$$

- (Lee Ch 8 # 3) A criterion for functions to be coordinate charts. You previously did something like this with Lee's Chapter 2 #17.

Solution: Let $x_1, \dots, x_m$ be coordinates in a neighborhood of p, so $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ is a basis for $T_p(M)$, and so $df_1 \wedge \cdots \wedge df_n(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ is nonzero. Setting $J = \left(df_i(\frac{\partial}{\partial x_j})\right) = \left(\frac{\partial f_i}{\partial x_j}\right)$,

$$\det J = df_1 \wedge \cdots \wedge df_n(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}) \neq 0.$$

Since J is nonsingular, the inverse function theorem gives a neighborhood V of p where $f = (f_1, \dots, f_n)$ is a diffeomorphism to a subset of $\mathbb{R}^n$, i.e. a coordinate chart.

- Give a natural geometric criterion for a basis $\mathbf{u}, \mathbf{v}$ of $\mathbb{R}^2$ to be positively oriented.

Solution: Write $\mathbf{u} = |u|(\cos\theta, \sin\theta)$ and $\mathbf{v} = |v|(\cos\phi, \sin\phi)$ for $\theta, \phi \in [0, 2\pi]$. The basis $\mathbf{u}, \mathbf{v}$ is positively oriented if $\det(\mathbf{u}|\mathbf{v}) > 0$. Now

$$\det(\mathbf{u}|\mathbf{v}) = |u||v|\cos\theta\sin\phi - \sin\theta\cos\phi = |u||v|\sin(\phi - \theta).$$

Now $\phi - \theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$ measured clockwise from $\mathbf{u}$ to $\mathbf{v}$. It's sine will be positive when it is an acute angle. In other words, a basis in $\mathbb{R}^2$ is positively oriented if the clockwise angle from its first vector to its second vector is acute.

- For which n is $\mathbb{R}P^n$ orientable?

Solution: Let Ω be the orientation of S^n as a submanifold of $\mathbb{R}^{n+1}$ given by

$$\Omega_p(X_1, \ldots, X_n) = \det(p|X_1|\cdots|X_n)$$

or equivalently

$$\Omega = \sum_{i=1}^{n} x_i dx_1 \wedge \cdots \hat{dx_i} \cdots \wedge dx_n$$

The antipodal map $a(x_1, \ldots, x_n) = (-x_1, \ldots, -x_n)$ acts on Ω as

$$\begin{aligned}(a^*\Omega)_p(X_1, \ldots, X_n) &= \Omega_{-p}(-X_1, \ldots, -X_n) = \det(-p| - X_1|\cdots| - X_n) \\ &= (-1)^{n+1}\Omega_p(X_1, \ldots, X_n)\end{aligned}$$

So $a^*\Omega = \Omega$ when n is odd, and $a^*\Omega = -\Omega$ when n is even.

When n is odd, $\mathbb{R}P^n$ is orientable. The orientation is given by pushing Ω down to $\mathbb{R}P^n$. More precisely, if $\pi : S^n \to \mathbb{R}P^n$, define

$$(\pi_*\Omega)_p(X_1, \ldots, X_n) = \Omega_{\tilde{p}}(\tilde{X}_1, \ldots, \tilde{X}_n)$$

for either $\tilde{p}$ with $\pi(\tilde{p}) = p$, and $T\pi(\tilde{X}_i) = X_i \in T_p\mathbb{R}P^n$. This is well defined since the antipodal map preserves Ω.

When n is even, $\mathbb{R}P^n$ is not orientable. Suppose it is. Then there is an orientation given by a nonvanishing $\Phi \in \bigwedge^n(\mathbb{R}P^n)$. Then $\tilde{\Phi} = \pi^*\Phi$ is a non-vanishing n-form on S^n, and so $\tilde{\Phi}_p = \lambda(p)\Omega_p$ for some non-vanishing function $\lambda \in C^\infty(S^n)$. Apply the antipodal map to this, so $\tilde{\Phi}_p = \lambda(-p)(-\Omega_p)$, so that $\lambda(p) = -\lambda(-p)$, which is impossible for a nonvanishing continuous function on the connected manifold S^n.