Week 6 Exercises

1. Suppose M is a manifold and g, h are both Riemannian metrics on M. Show that for any $p \in M$, there is a neighborhood U of p and scalars $\lambda, \mu > 0$ so that for any $x \in U$ and any $v \in T_x M$, we have:
$$\lambda g(v,v) \leq h(v,v) \leq \mu g(v,v).$$

2. Let ω be an arbitrary smooth 1-form on M. For vector fields $X, Y \in \mathfrak{X}(M)$, define $\Omega(X,Y) = \omega([X,Y])$. Show that Ω is bilinear and anti-symmetric. Is Ω a tensor?

3. Given intervals I, J, and unit speed curves $f : I \to M$, $g : J \to M$, let $\Gamma \subset I \times J$ be the set of (s,t) with $f(s) = g(t)$. Because $g^{-1} \circ f$ is a diffeomorphism on its domain, Γ consists of line segments of slope ± 1 which must extend to the boundary of $I \times J$, and at most one of these segments can end on a given edge of the rectangle $I \times J$. Explain why Γ has at most two components, and draw all combinatorial possibilities for Γ.