Week 6 Exercises

1. Suppose M is a manifold and g, h are both Riemannian metrics on M. Show that for any $p \in M$, there is a neighborhood U of p and scalars $\lambda, \mu > 0$ so that for any $x \in U$ and any $v \in T_xM$, we have:
$$\lambda g(v,v) \leq h(v,v) \leq \mu g(v,v).$$

Solution: Let V be any neighborhood of p, and let $x \in V$. The set $S_x = \{u \in T_xM | g(u,u)\}$ is a sphere in the inner product space (T_xM, g), so S_x is compact. Then the continuous function $h(u,u)$ for $u \in S_x$ has both a maximum μ_x and minimum λ_x. Now for any nonzero $v \in T_xM$, put $u = v/\sqrt{g(v,v)}$. Then

$$\begin{aligned} \lambda_x &\leq h(u,u) \leq \mu_x \\ \text{so } \lambda_x &\leq \frac{h(v,v)}{g(v,v)} \leq \mu_x \\ \text{so } \lambda_x g(v,v) &\leq h(v,v) \leq \mu_x g(v,v). \end{aligned}$$

Now this holds for any $v \in T_xM$. The values λ_x, μ_x depend continuously on x. So, let U be a neighborhood of p with $\bar{U} \subset V$ and $\bar{U}$ compact. Let $\lambda = \min_{x \in \bar{U}} \lambda_x$ and $\mu = \max x \in \bar{U} \mu_x$. Then for any $x \in U$ and any $v \in T_xM$,

$$\lambda g(v,v) \leq \lambda_x g(v,v) \leq h(v,v) \leq \mu_x g(v,v) \leq \mu g(v,v).$$

2. Let ω be an arbitrary smooth 1-form on M. For vector fields $X, Y \in \mathfrak{X}(M)$, define $\Omega(X,Y) = \omega([X,Y])$. Show that Ω is bilinear and anti-symmetric. Is Ω a tensor?

Solution:

$$\Omega(X,Y) = \omega([X,Y]) = \omega(-[Y,X]) = -\omega([Y,X]) = -\Omega(Y,X)$$

so that Ω is antisymmetric. For $X_1, X_2 \in \mathfrak{X}(M)$ and c a scalar,

$$\begin{aligned} \Omega(cX_1 + X_2, Y) = \omega([cX_1 + X_2, Y]) &= \omega(cX_1Y + X_2Y - YcX_1 - YX_2) = \\ &= \omega(c[X_1,Y] + [X_2,Y]) = c\omega([X_1,Y]) + \omega([X_2,Y]) = \\ &= c\Omega(X_1,Y) + \Omega(X_2,Y). \end{aligned}$$

This shows linearity in the first argument, and antisymmetry implies linearity in the second argument. So Ω is bilinear.

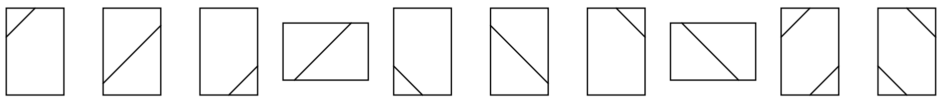
However, Ω is not generally linear over $C^\infty(M)$. If $f \in C^\infty(M)$, then

$$\begin{aligned} \Omega(fX,Y) &= \omega([fX,Y]) = \omega(fXY - Y(fX)) = \omega(fXY - (Yf)X - fYX) \\ &= \omega(f[X,Y] - (Yf)X) = f\Omega(X,Y) - (Yf)\omega(X). \end{aligned}$$

If we choose X so $\omega(X) \neq 0$, and choose Y and f so $Yf \neq 0$, then $\Omega(fX,Y) \neq f\Omega(X,Y)$, and Ω is not a tensor.

3. Given intervals I, J, and unit speed curves $f : I \to M$, $g : J \to M$, let $\Gamma \subset I \times J$ be the set of (s, t) with $f(s) = g(t)$. Because $g^{-1} \circ f$ is a diffeomorphism on its domain, Γ consists of line segments of slope ± 1 which must extend to the boundary of $I \times J$, and at most one of these segments can end on a given edge of the rectangle $I \times J$. Explain why Γ has at most two components, and draw all combinatorial possibilities for Γ.

Solution: Each segment has two ends, and each end needs to hit one edge of the rectangle. Since the rectangle has four edges, there can be at most two segments. There are ten ways this can happen:



One might consider cases where the segments hit precisely at the corners of the rectangle. Combinatorically these are different, but are just special cases of the ten above when it comes to the proof that uses this list.