Week 5 Projects

- Boothby pg 187 #2: Show there is a correspondence:

  $$\text{fields of bilinear forms on } M \longleftrightarrow C^\infty(M)\text{-bilinear mappings } \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

  Hints: The $\to$ direction is easy. In the other direction, you have a bilinear mapping of vector fields on M, and given $v, w \in T_p(M)$, you need to define $\Phi_p(v, w)$. Do this by extending v, w to vector fields, and then proving the definition is independent of the extensions chosen. The key step is to use local coordinates near p. If two extensions agree at p, then their coefficients in coordinates agree at p. Applying a cutoff function, you can extend the coefficients to all of M and pull them out of Φ.

  Another big hint is that this is the specific case of Lee's Proposition 7.32, with $r = 0$ and $s = 2$.

- Boothby Theorem (3.1). A connected Riemannian manifold is a metric space, with the metric

  $$d(p, q) = \inf_\gamma \text{length}(\gamma)$$

  where the infimum is over all piecewise C^1 curves joining $p, q \in M$.

  Hints: Essentially the same proof is in Gallot-Hulin-Lafontaine, pg 87-89. Both arguments reduce the theorem to the same the basic fact: a straight line is the shortest curve between two points in $\mathbb{R}^n$. This basic fact is Exercises 5 and 6 of Boothby, pg. 192, but you may assume it.

- The classification of 1-dimensional manifolds. This is shown in Appendix A of Milnor, *Topology from the Differentiable Viewpoint*. The argument is elementary, but does require the existence of a Riemannian metric so that curves can be parameterized by arc length.