Math 642

Week 4 Exercises

- Boothby pg 187 #1 is a straightforward warm-up problem with basic linear algebra facts about bilinear forms.
- Boothby pg 187 #2: Show there is a correspondence:

  $$\text{fields of bilinear forms on } M \longleftrightarrow C^\infty(M)\text{-bilinear mappings } \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

  Hints: The $\to$ direction is easy. In the other direction, you have a bilinear mapping of vector fields on M, and given $v, w \in T_p(M)$, you need to define $\Phi_p(v, w)$. Do this by extending v, w to vector fields, and then proving the definition is independent of the extensions chosen. The key step is to use local coordinates near p. If two extensions agree at p, then their coefficients in coordinates agree at p. Applying a cutoff function, you can extend the coefficients to all of M and pull them out of Φ.

  Another big hint is that this is the specific case of Lee's Proposition 7.32, with $r = 0$ and $s = 2$.

- Boothby pg 192 # 3 (definition of the gradient)

1. Given $c(u) = (r(u), z(u))$ a smooth curve in the x-z plane with $r(u) \neq 0$, let $M \subset \mathbb{R}^3$ be the surface of revolution of c around the z-axis. Find the metric g on M as a submanifold of $\mathbb{R}^3$.

- Boothby pg 192 # 2 (the metric on the torus in $\mathbb{R}^3$) You might apply the previous problem.