Week 4 Exercises

- Boothby pg 187 #1 is a straightforward warm-up problem with basic linear algebra facts about bilinear forms.

Solution: For (iv), write:

$$\Phi(v,w) = \frac{1}{2}\left(\Phi(v,w) + \Phi(w,v)\right) + \frac{1}{2}\left(\Phi(v,w) - \Phi(w,v)\right)$$

For (v), choose a basis $e_1, \dots, e_n$ and put $\Phi(e_i, e_j) = A_{ij}$. Since $A_{ji} = \Phi(e_j, e_i) = -\Phi(e_i, e_j) = -A_{ij}$, the matrix A is skew symmetric. Then $\det(A) = \det(-A^t) = (-1)^n \det(A)$. When n is odd, we have $2\det(A) = 0$ so A must be singular, and Φ does not have rank n.

- Boothby pg 187 #2: Show there is a correspondence:

$$\text{fields of bilinear forms on } M \longleftrightarrow C^\infty(M)\text{-bilinear mappings } \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

Hints: The $\to$ direction is easy. In the other direction, you have a bilinear mapping of vector fields on M, and given $v, w \in T_p(M)$, you need to define $\Phi_p(v,w)$. Do this by extending v, w to vector fields, and then proving the definition is independent of the extensions chosen. The key step is to use local coordinates near p. If two extensions agree at p, then their coefficients in coordinates agree at p. Applying a cutoff function, you can extend the coefficients to all of M and pull them out of Φ.

Another big hint is that this is the specific case of Lee's Proposition 7.32, with $r = 0$ and $s = 2$.

Solution:

$\to$: Given a bilinear form Ψ on M and $X, Y \in \mathfrak{X}(M)$, define $\Phi(X,Y) \in C^\infty(M)$ by $\Phi(X,Y)(p) = \Psi_p(X_p, Y_p)$. This is bilinear over $C^\infty(M)$: if $f \in C^\infty(M)$, then

$$\Phi(fX, Y)(p) = \Psi_p(f(p)X_p, Y_p) = f(p)\Psi_p(X_p, Y_p) = f(p)\Phi(X,Y)(p),$$

and similarly $\Phi(X, fY) = f\Phi(X,Y)$.

$\leftarrow$: Let Φ be a $C^\infty(M)$-bilinear mapping $\mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$. Fix $p \in M$. For $v, w \in T_p(M)$, let $V, W \in \mathfrak{X}(M)$ be C^∞-vector fields on M with $V_p = v$, $W_p = w$ (we've proved these exist using cutoff functions). Define $\Psi_p(v,w) = \Phi(V,W)(p)$. The difficulty here is to show that this definition is independent of the extending vector fields.

So, let $\bar{V}, \bar{W}$ be any other vector fields with $\bar{V}_p = v$ and $\bar{W}_p = w$. We will show that $\Phi(X,Y)(p) = 0$ whenever $X_p = 0$ or $Y_p = 0$, so that

$$\begin{aligned}
\Phi(V,W)(p) - \Phi(\bar{V},\bar{W})(p) &= \Phi(V - \bar{V}, W)(p) + \Phi(\bar{V}, W)(p) && (1)\\
&\quad - \Phi(\bar{V}, W)(p) + \Phi(\bar{V}, W - \bar{W})(p) && (2)\\
&= \Phi(V - \bar{V}, W)(p) + \Phi(\bar{V}, W - \bar{W})(p) && (3)\\
&= \Phi(X, W)(p) + \Phi(\bar{V}, Y) = 0. && (4)
\end{aligned}$$

with $X = V - \bar{V}, Y = W - \bar{W}$.

This leaves the clam that $\Phi(X,Y)(p) = 0$ when $X_p = 0$ or $Y_p = 0$. The two cases are similar, so suppose $X_p = 0$. Choose coordinates $x_1, \dots x_n$ on a neighborhood U of p, and write $X = \sum_i \xi_i \frac{\partial}{\partial x_i}$, with each $\xi_i(p) = 0$. Let β be a smooth cutoff function which is 1 at p and vanishes outside of U. Then $\beta \frac{\partial}{\partial x_i}$ are vector fields on all of M, and $\beta \xi_i$ is a smooth function on all of M. So (using $C^\infty(M)$ bilinearity of Φ):

$$\beta^2 \Phi(X,Y) = \Phi(\sum_i \beta \xi_i \beta \frac{\partial}{\partial x_i}, Y) = \sum_i \beta \xi_i \Phi(\beta \frac{\partial}{\partial x_i}, Y)$$

Evaluting at p, the left hand side is $\Phi(X,Y)(p)$ and the right hand side is 0.

- Boothby pg 192 # 3 (definition of the gradient)

Solution: Let $\langle \cdot, \cdot \rangle$ denote the Riemannian metric on M. Given $X \in \mathfrak{X}(M)$, define $\sigma_X(Y_p) = \langle X_p, Y_p \rangle$ for $Y_p \in T_p(M)$. If $X' \in \mathfrak{X}(M)$ and $c \in \mathbb{R}$,

$$\sigma_{X+cX'}(Y_p) = \langle X_p + cX'_p, Y_p \rangle = \langle X_p, Y_p \rangle + c \langle X'_p, Y_p \rangle = \sigma_X(Y_p) + c\sigma_{X'}(Y_p)$$

so that the map $X \to \sigma_X$ is linear. An identical argument with c replaced by f shows that it is $C^\infty(M)$ linear.

Now given $f \in C^\infty(M)$, df is a one-form. By (iii) of problem 1 above, there is a unique $X_p \in T_p(M)$ so that $\langle X_p, Y_p \rangle = df(Y_p)$ for all $Y_p \in T_p(M)$. If we can show X is smooth, then $df = \sigma_X$ and $\operatorname{grad} f := X$.

I don't see any great way to show $\operatorname{grad} f$ is smooth except by computing it in local coordinates:

$$\left\langle \operatorname{grad} f, \frac{\partial}{\partial x_i} \right\rangle = df(\frac{\partial}{\partial x_i}) = \frac{\partial f}{\partial x_i}.$$

If g_{ij} are the coefficients of the metric $\langle \cdot, \cdot \rangle$ and $\operatorname{grad} f = \sum_i a_i \frac{\partial}{\partial x_i}$, then

$$\left\langle \operatorname{grad} f, \frac{\partial}{\partial x_i} \right\rangle = \sum_j a_j g_{ij}$$

so that

$$a_j = \sum_i \frac{\partial f}{\partial x_i} g^{ij}$$

where g^{ij} is the inverse of the matrix g_{ij}. Thus a_i is smooth, and so $\operatorname{grad} f$ is smooth.

It's worth noting that if $\frac{\partial}{\partial x_i}$ are orthonormal, then g is the identity matrix and $X = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m})$.

1. Given $c(u) = (r(u), z(u))$ a smooth curve in the x-z plane with $r(u) \neq 0$, let $M \subset \mathbb{R}^3$ be the surface of revolution of c around the z-axis. Find the metric g on M as a submanifold of $\mathbb{R}^3$.

Solution: Parameterize by (u, θ) with $x = r(u)\cos\theta, y = r(u)\sin\theta, z = z(u)$. Then

$$dx = r'(u)\cos\theta du - r(u)\sin\theta d\theta \tag{5}$$
$$dx = r'(u)\sin\theta du + r(u)\cos\theta d\theta \tag{6}$$
$$dz = z'(u)du \tag{7}$$

Then

$$dx^2 + dy^2 + dz^2 = (r'(u)^2 + z'(u)^2)du^2 + r(u)^2 d\theta^2 = |c'(u)|^2 du^2 + r(u)^2 d\theta^2.$$

- Boothby pg 192 # 2 (the metric on the torus in $\mathbb{R}^3$) You might apply the previous problem.

Solution: This is the surface of revolution of the curve $c(\varphi) = (a + b\cos\varphi, b\sin\varphi)$ around the z-axis. Applying the previous problem, the metric is

$$b^2 d\varphi^2 + (a + b\cos\varphi)^2 d\theta^2.$$