Week 3 Exercises

(Corrected problem 2, added #4.)

1. Let $\omega = \frac{1}{x^2+y^2}(-ydx + xdy)$ on $M = \mathbb{R}^2 - \{0\}$. Let $\theta \in (a, 2\pi + a)$, $r \in (0, \infty)$, and R be the ray from the origin at angle a. Then (r, θ) give polar coordinates on $\mathbb{R}^2 - R$.
   (a) Show that $\omega = d\theta$ on $\mathbb{R}^2 - R$.
   (b) Let γ be the closed curve $\gamma(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$. Compute $\int_\gamma \omega$.
   (c) Is ω exact? Is ω conservative? Is ω locally conservative?

2. Let σ be a locally conservative 1-form on $M = \mathbb{R}^2 - \{0\}$.
   (a) Show that σ is exact if and only if $\int_c \sigma = 0$, where c is the curve that goes around the unit circle once, clockwise.
   (b) Show that *any* locally conservative one-form σ on $\mathbb{R}^2 - \{0\}$ can be written as $\sigma = \lambda\omega + df$, where ω is as in problem 1, $\lambda \in \mathbb{R}$, and $f \in C^\infty(M)$.

3. Consider the two dimensional torus $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(n, m) \cdot (x, y) = (x + n, y + m)$. Define one forms σ and τ on M by $\sigma(v) = dx(\tilde{v})$, $\tau(v) = dy(\tilde{v})$ for $v \in TM$ and $\tilde{v}$ is any lift of v to $T\mathbb{R}^2$. Show that σ and τ are well defined and locally conservative, but not exact.

4. Boothby, Pg. 187 #9: Show that $\text{tr}(A^T B)$ defines a symmetric bilinear form on $M_n(\mathbb{R})$.