Week 3 Exercises

(Corrected problem 2, added #4.)

1. Let $\omega = \frac{1}{x^2+y^2}(-ydx + xdy)$ on $M = \mathbb{R}^2 - \{0\}$. Let $\theta \in (a, 2\pi + a)$, $r \in (0, \infty)$, and R be the ray from the origin at angle a. Then (r, θ) give polar coordinates on $\mathbb{R}^2 - R$.
   (a) Show that $\omega = d\theta$ on $\mathbb{R}^2 - R$.
   (b) Let γ be the closed curve $\gamma(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$. Compute $\int_\gamma \omega$.
   (c) Is ω exact? Is ω conservative? Is ω locally conservative?

Solution:

(a) We have $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$dx = \cos\theta dr - r\sin\theta d\theta, \quad dy = \sin\theta dr + r\cos\theta d\theta.$$

Substituting these into the formula for ω shows $\omega = d\theta$.

(b) Calculate $\gamma'(t) = (-\sin(t), \cos(t))$ so $\omega(\gamma'(t)) = \sin^2(t) + \cos^2(t) = 1$, so

$$\int_\gamma \omega = \int_0^{2\pi} 1dt = 2\pi.$$

(c) If ω were conservative, the integral in part (b) would vanish. So ω is not conservative, and therefore not exact. It is, however, locally conservative since any $p \in M$ has a neighborhood of the form $\mathbb{R}^2 - R$ and $\omega = d\theta$ on that neighborhood.

2. Let σ be a locally conservative 1-form on $M = \mathbb{R}^2 - \{0\}$.
   (a) Show that σ is exact if and only if $\int_c \sigma = 0$, where c is the curve that goes around the unit circle once, clockwise.
   (b) Show that *any* locally conservative one-form σ on $\mathbb{R}^2 - \{0\}$ can be written as $\sigma = \lambda\omega + df$, where ω is as in problem 1, $\lambda \in \mathbb{R}$, and $f \in C^\infty(M)$.

Solution:

(a) If σ is exact, then any integral around a closed curve is 0, so $\int_c \sigma = 0$. Now suppse $\int_c \sigma = 0$. Let γ be any closed curve in M. Since the fundamental group $\pi_1(M) = \mathbb{Z}$ is generated by c, γ is homotopic to c^k for some k. Since σ is locally conservative, line integrals are homotopy invariant, and

$$\int_\gamma \sigma = \int_{c^k} \sigma = k\int_c \sigma = 0.$$

Then σ is conservative, and therefore exact.

(b) Let

$$\lambda = \frac{1}{2\pi}\int_c \sigma.$$

Then

$$\int_c \sigma - \lambda\omega = \int_c \sigma - \lambda \int_c \omega = 2\pi\lambda - \lambda 2\pi = 0.$$

Now $\sigma - \lambda\omega$ is locally conservative (since both σ and ω are), so by part (a), there is some f with $\sigma - \lambda\omega = df$, so $\sigma = \lambda\omega + df$.

3. Consider the two dimensional torus $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(n,m)\cdot(x,y) = (x+n, y+m)$. Define one forms σ and τ on M by $\sigma(v) = dx(\tilde{v})$, $\tau(v) = dy(\tilde{v})$ for $v \in TM$ and $\tilde{v}$ is any lift of v to $T\mathbb{R}^2$. Show that σ and τ are well defined and locally conservative, but not exact.

Solution: Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the projection map. Given $v \in T_p\mathbb{T}^2$, let $\tilde{v} \in T_{\tilde{p}}\mathbb{R}^2$ be a lift of v, so $T\pi(\tilde{v}) = v$. Write $\tilde{v} = v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}$, so that $\sigma(v) = v_1$, $\tau(v) = v_2$. Any other lift of v is related to $\tilde{v}$ by a diffeomorphism $g : (x,y) \to (x+n, y+m)$ for some $m, n \in \mathbb{Z}$. Since $Tg = I$, $Tg(\tilde{v}) = v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} \in T_{g(\tilde{p})}\mathbb{R}^2$, so $dx(Tg(\tilde{v})) = v_1$ and $dy(Tg(\tilde{v})) = v_2$, which shows that σ, τ are well defined. This sort of argument works more generally for a group action on a manifold. The important fact is that the forms dx and dy are invariant under the group action.

Next, see that σ, τ are locally conservative. The point here is that in a small neighborhood of a point, closed curves on $\mathbb{T}^2$ lift to closed curves in $\mathbb{R}^2$ where dx and dy are exact. For $p \in \mathbb{T}^2$, choose $\tilde{p} \in \mathbb{R}^2$ with $\pi(\tilde{p}) = p$. There is a neighborhood $\tilde{U}$ of $\tilde{p}$ on which π is a diffeomorphism (this is true in general for covering maps, here $\tilde{U}$ could be an open ball of radius 1/2). Let $U = \pi(\tilde{U})$. For any closed curve $c \subset U$, $\tilde{c}(t) = \pi^{-1}(c(t))$ defines a closed curve in $\tilde{U}$, and

$$\int_c \sigma = \int_{\tilde{c}} dx = 0$$

since dx is exact on $\mathbb{R}^2$. The argument for τ is the same, with dy.

Let c be the curve $c(t) = (t, 0) \in \mathbb{R}^2$. Since $c(t+1) = (1,0)\cdot c(t)$, c defines a closed curve on $\mathbb{T}^2$ for $t \in [0,1]$, and

$$\int_c \sigma = \int_0^1 \sigma(c'(t))dt = \int_0^1 dx(\frac{\partial}{\partial x})dt = \int_0^1 1dt = 1$$

which shows that σ is not exact. Use $c(t) = (0, t)$ to show τ is not exact.

4. Boothby, Pg. 187 #9: Show that $\Phi(A,B) = \mathrm{tr}(A^T B)$ defines a symmetric bilinear form on $M_n(\mathbb{R})$.

Solution: First, Φ is symmetric since

$$\Phi(A,B) = \mathrm{tr}(A^T B) = \mathrm{tr}((A^T B)^T) = \mathrm{tr}(B^T A^{TT}) = \mathrm{tr}(B^T A) = \Phi(B,A).$$

Next, Φ is linear in the first argument since for matrices A and A' and scalar c,

$$\Phi(A + cA', B) = \operatorname{tr}((A + cA')^T B) = \operatorname{tr}(A^T B) + c \operatorname{tr}(A'B) = \Phi(A, B) + c\Phi(A', B).$$

Linearity in the second argument follows by symmetry. Φ is positive definite. One way to see this is with matrix entries, where $\operatorname{tr}(A^T A) = \sum_{i,j} a_{ij}^2 \geq 0$ with equality if and only if A is the zero matrix. Or, notice that $A^T A$ is symmetric, hence has real eigenvalues $\lambda_1, \ldots, \lambda_n$ with eigenvectors $v_1, \ldots, v_n$. Then, using the usual inner product on $\mathbb{R}^n$,

$$\left\langle A^T A v_i, v_i \right\rangle = \langle A v_i, A v_i \rangle \geq 0$$

while

$$\left\langle A^T A v_i, v_i \right\rangle = \langle \lambda_i v_i, v_i \rangle = \lambda_i \langle v_i, v_i \rangle$$

so that $\lambda_i = \langle A v_i, A v_i \rangle / \langle v_i, v_i \rangle \geq 0$. Then $\operatorname{tr}(A^T A) = \lambda_1 + \cdots + \lambda_n \geq 0$ with equality if and only if all eigenvalues of A are 0, which means A is the zero matrix.