Week 2 Exercises

- Boothby pg. 183 #5: Determine the subset of $\mathbb{R}^2$ on which $\sigma_1 = x\,dx + y\,dy$ and $\sigma_2 = y\,dx + x\,dy$ are linearly independent and find a frame field dual to σ_1, σ_2 on this set.

Solution: They are independent when $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ is non-singular. The determinant is $x^2 - y^2$, so they are linearly independent on $\{(x,y) | x^2 - y^2 \neq 0\}$. The dual frame field is described by the inverse matrix

$$\frac{1}{x^2 - y^2} \begin{pmatrix} x & -y \\ -y & x \end{pmatrix}$$

and given by $X_1 = (x^2 - y^2)^{-1} \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$ and $X_2 = (x^2 - y^2)^{-1} \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right)$

- Boothby pg. 183 #6: Show that the restriction of $\sigma = x\,dy - y\,dx + z\,dw - w\,dz$ from $\mathbb{R}^4$ to the sphere S^3 is never zero on S^3.

Solution: Let $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}$. V is a vector field on S^3, since it is perpendicular to the radial field $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}$. Then compute $\sigma(V) = x^2 + y^2 + z^2 + w^2 = 1$, so σ never vanishes on S^3.

- Lee Ch 2 Problem 13 (Hessian). Remark: When $df = 0$, f has a critical point, the 'second derivative test' determines the behavior near the critical point using the eigenvalues of the Hessian. For example, in $\mathbb{R}^2$, try the functions $x^2 + y^2$, $x^2 - y^2$, $-x^2 - y^2$, and xy at the origin.

Solution: Since $df_p = 0$, $X_p Y f - Y_p X f = [X, Y]_p f = df_p([X, Y]_p) = 0$. It is apparent that $H(v, w) = X_p(Y f)$ only depends on $v = X_p$, and not the extension vector field X, and is linear in $v = X_p$. Since $X_p(Y f) = Y_p(X f)$, the Hessian $H(v, w)$ depends only on w and not the extension to Y, and is linear in w.

Suppose $df_p \neq 0$. The claim is that $H(v, w) = X_p(Y f)$ is not well defined - it depends on the choice of Y. Choose $w \in T_p M$ so that $df_p(w) = 1$. Let Y be a vector field on M with $Y_p = w$. Let v and X be arbitrary with $X_p = v$. Let g be any smooth function on M with $g(p) = 1$. Then gY is another extension of w, and compute:

$$X_p(gY f) = (X_p g)(Y_p f) + g(p) X_p Y f = X_p g(df(Y_p)) + X_p Y f = X_p g + X_p Y f.$$

So, choosing a g with $X_p g \neq 0$, we see that $H(v, w)$ as defined is dependent on the extension of w to Y.

- Lee Ch 2 Problem 17 (Test for coordinate charts).

Solution: Since $df_1(p), \ldots, df_N(p)$ spans $T_p^*(M)$, we can discard some of the f_i and renumber them so that $df_1(p), \ldots, df_m(p)$ form a basis for $T_p^*(M)$. Let $x_1, \ldots, x_m$ be coordinates in a neighborhood of p, so $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}$ is a basis for $T_p(M)$. Then the matrix $J = \left(df_i(\frac{\partial}{\partial x_j})\right) = \left(\frac{\partial f_i}{\partial x_j}\right)$ is nonsingular, so by the inverse function theorem, there is a neighborhood V of p where $f = (f_1, \ldots, f_m)$ is a diffeomorphism, i.e. a coordinate system on V.