Week 14 Exercises

1. For $n \in \mathbb{Z}$, let P_n be the 0-manifold consisting of n points $p_1, \dots, p_n$. Suppose $f : P_n \to P_m$ is any map (obviously smooth).
   (a) Using the natural basis for H^0 describe the matrix $f^* : H^0(P_m) \to H^0(P_n)$.
   (b) When is f^* injective? When is f^* surjective?

Solution:

(a) The basis elements for $H^0(P_m)$ are $\delta_1, \dots, \delta_m$, where $\delta_i(p_j) = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$. Then $f^*\delta_i = \delta_i \circ f$, which is 1 on points q with $f(q) = p_i$ and 0 otherwise. The matrix for f^* has m columns, and column i has a one in every row j such that $f(p_j) = p_i$, and zeros elsewhere.

(b) f^* is injective when f is surjective, and f^* is surjective when f is injective.

2. Suppose M is a manifold with finite fundamental group. Show that $H^1(M) = 0$.
   Hint: The universal cover $\tilde{M}$ of M is simply connected, and $\pi_1(M)$ is a finite group of diffeomorphisms of $\tilde{M}$.

Solution: Let $\Pi : \tilde{M} \to M$ be the universal cover of M. Let $\omega \in \bigwedge^1(M)$ be any closed one-form. Let $\tilde{\omega} = \Pi^*\omega$ be the lift of ω to $\tilde{M}$.

Now $d\tilde{\omega} = d\Pi^*\omega = \Pi^* d\omega = 0$, so $\tilde{\omega}$ is closed. Since $\tilde{M}$ is simply connected, $H^1(\tilde{M}) = 0$ and so $\tilde{\omega}$ is exact, say $\tilde{\omega} = d\tau$ for some $\tau \in C^\infty(\tilde{M})$.

The finite group $\pi_1(M)$ is a group of diffeomorphisms of $\tilde{M}$. Now let

$$\tilde{\sigma} = \frac{1}{|\pi_1(M)|} \sum_{g \in \pi_1(M)} g^*\tau.$$

We have

$$\begin{aligned} d\tilde{\sigma} &= \frac{1}{|\pi_1(M)|} \sum_{g \in \pi_1(M)} dg^*\tau \\ &= \frac{1}{|\pi_1(M)|} \sum_{g \in \pi_1(M)} g^* d\tau \\ &= \frac{1}{|\pi_1(M)|} \sum_{g \in \pi_1(M)} g^*\tilde{\omega} = \tilde{\omega} \end{aligned}$$

where the last step uses the fact that $\tilde{\omega}$ is a lift of ω, and is therefore invariant under the action of $\pi_1(M)$.

Then $\tilde{\sigma}$ is invariant under the action of π_1, and so is a lift of a function $\sigma \in C^\infty(M)$.

Finally, $\Pi^* d\sigma = d\Pi^*\sigma = d\tilde{\sigma} = \tilde{\omega} = \Pi^*\omega$. Since Π is surjective, Π^* is injective and we get $d\sigma = \omega$, so that ω is exact. This shows every closed one-form is exact, so $H^1(M) = 0$.

3. Let $M = S^1$ with coordinate θ, and let t be the coordinate on $\mathbb{R}$. Define a (not particularly special) one-form on $M \times \mathbb{R}$ by:

$$\omega = t \cos\theta dt + t^2 \sin\theta d\theta.$$

With K and s_a as in Lee, Theorem 10.8, check that $(id - \pi^* s_a^*)\omega = (dK - Kd)\omega$.

Solution: First, $s_a^*\omega = a^2 \sin\theta d\theta$, so that

$$(id - \pi^* s_a^*)\omega = t\cos\theta dt + (t^2 - a^2)\sin\theta d\theta.$$

Next, compute $d\omega = 3t\sin\theta dt \wedge d\theta$ and $\iota_{\partial/\partial t} d\omega = 3t \sin\theta d\theta$, and

$$Kd\omega = (-1)^{2-1}\int_{\tau=a}^{t} 3\tau \sin\theta d\theta d\tau = -\frac{3}{2}(t^2 - a^2)\sin\theta d\theta.$$

On the other hand, $\iota_{\partial/\partial t}\omega = t\cos\theta$, so

$$K\omega = (-1)^{1-1}\int_{\tau=a}^{t} \tau\cos\theta d\tau = \frac{1}{2}(t^2 - a^2)\cos\theta,$$

and

$$dK\omega = t\cos\theta dt - \frac{1}{2}(t^2 - a^2)\sin\theta d\theta$$

so in fact, $(id - \pi^* s_a^*)\omega = (dK - Kd)\omega$.

4. Define the cup product $\cup : H^k(M) \times H^\ell(M) \to H^{k+\ell}(M)$ by

$$[\alpha] \cup [\beta] = [\alpha \wedge \beta].$$

Show $\cup$ is well defined and bilinear.

Solution: Suppose $[\alpha'] = [\alpha]$ and $[\beta'] = [\beta]$. Write $\alpha' = \alpha + d\sigma$ and $\beta' = \beta + d\tau$. Since α and β are closed,

$$d(\sigma \wedge \beta) = d\sigma \wedge \beta \quad \text{and} \quad d(\alpha \wedge \tau) = (-1)^k \alpha \wedge d\tau$$

Then

$$\begin{aligned} \alpha' \wedge \beta' &= (\alpha + d\sigma) \wedge (\beta + d\tau) \\ &= \alpha \wedge \beta + d\sigma \wedge \beta + \alpha \wedge d\tau + d\sigma \wedge d\tau \\ &= \alpha \wedge \beta + d(\sigma \wedge \beta) + d((-1)^k \alpha \wedge \tau) + d(\sigma \wedge d\tau) \end{aligned}$$

Then $[\alpha' \wedge \beta'] = [\alpha \wedge \beta]$ and the cup product is well defined. Bilinearity follows trivially from bilinearity of wedge product.