Week 13 Exercises

- Lee Ch 9 Problem 6ab (but not the extra part a, which is not part of problem 6). This problem should say that M has no boundary.

Solution:

(a) Following the definition of δ in Lee 9.6, $\Delta = \delta d = - * d * d$. If f is a constant function, $\Delta f = \delta df = 0$, so 0 is an eigenvalue of Δ. Then any eigenfunction f satisfies $\delta df = \lambda f$, so that $\lambda(f|f) = (\lambda f, f) = (\delta df|f) = (df|df) \geq 0$, and since $(f|f) > 0$, $\lambda \geq 0$.

(b) Using the self adjoint property of Δ:

$$\lambda_1(f_1|f_2) = (\lambda_1 f_1|f_2) = (\Delta f_1, f_2) = (f_1|\Delta f_2) = (f_1|\lambda_2 f_2) = \lambda_2(f_1|f_2).$$

If $(f_1|f_2) \neq 0$ then divide through and get $\lambda_1 = \lambda_2$.

A few words about manifolds with boundary here: First, if $M = [a, b]$ is an interval, then $\Delta = -\frac{\partial^2}{\partial x^2}$, and then $f(x) = \sin(ax)$ and $f(x) = e^{ax}$ are eigenfunctions with eigenvalues $\pm a^2$, so there really is a problem. It comes about because d and δ are no longer adjoint - instead there is a boundary correction term. In the proof of Proposition 9.45, if α and β are not zero on ∂M, then

$$\int_M d(\alpha \wedge *\beta) = \int_{\partial M} \alpha \wedge *\beta$$

is no longer zero. In part (a) above, this results in $(\delta df|f) = (df|df) - \int_{\partial M} df \wedge *f$, which should remind you of integration by parts.

To fix the problem, either assume M has no boundary, or else assume some sort of boundary conditions on the functions in question. The usual choices are that $f|_{\partial M} = 0$ (Dirichlet boundary conditions) or that $df|_{\partial M} = 0$ (Neumann boundary conditions). With either of these boundary conditions, d and δ are adjoint again, because $\int_{\partial M} df \wedge *f = 0$.

- Lee Ch 9 Problem 7. In this problem, it should say that N is a *unit* normal field.

Solution: For $p \in M$, let $X_1, \ldots, X_n$ be an orthonormal basis for $T_p(M)$, so that

$$\mathrm{vol}_M(X_1, \ldots, X_n) = 1.$$

If $\mu = dx^1 \wedge \cdots \wedge dx^{n+1}$ is the Euclidean volume form on $\mathbb{R}^{n+1}$, then the interior product $\iota_N \mu$ defines an n-form on M, and

$$\iota_N \mu(X_1, \ldots, X_n) = \mu(N, X_1, \ldots, X_n) = 1,$$

since $N, X_1, \ldots, X_n$ form a postively oriented orthonormal basis for $T_p\mathbb{R}^{n+1}$. Then $\mathrm{vol}_M = \iota_N \mu$ since the space of n-forms on $T_p(M)$ is one dimensional. Finally,

$$\mathrm{vol}_M = \iota_N(dx^1 \wedge \cdots \wedge dx^{n+1}) = \sum_i (-1)^{i-1} N^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}$$