Week 12 Exercises

1. (Lee Ch 9 Problem 2) Prove it, then interpret this particular case: Suppose $M = [a, b] \subset \mathbb{R}$ and f, g are functions on $[a, b]$. Check that the 0-forms $\alpha(x) = f(x) - f(a)$ and $\beta(x) = g(x) - g(b)$ satisfy the conditions of the problem. Simplify both sides until you find a familiar fact from calculus.

2. If M is a compact orientable manifold, show there is no smooth map $f : M \to \partial M$ with $f|_{\partial M} = Id$. Do we really need M to be compact? What if we only have ∂M compact?

   Hint: Integrate the volume form on ∂M in two ways: over ∂M and by pulling back to M.

3. Suppose M is a compact and orientable n-manifold with no boundary. Let $\theta \in \bigwedge^{n-1}(M)$ be any $n-1$ form. Show that $d\theta$ must vanish at some point of M.

4. Let (X, Y) be stereographic coordinates on S^2, with metric volume form

$$\mu = \frac{4}{(1 + X^2 + Y^2)^2} dX \wedge dY.$$

   Find the divergence Div $\frac{\partial}{\partial X}$.

   Where on the sphere is the divergence maximized and minimized? Make contour plots of Div $\frac{\partial}{\partial X}$ both in the planar (X, Y) coordinates and on the surface of the sphere. How does all this change for Div $\frac{\partial}{\partial Y}$?