Week 12 Exercises

1. (Lee Ch 9 Problem 2) Prove it, then interpret this particular case: Suppose $M = [a, b] \subset \mathbb{R}$ and f, g are functions on $[a, b]$. Check that the 0-forms $\alpha(x) = f(x) - f(a)$ and $\beta(x) = g(x) - g(b)$ satisfy the conditions of the problem. Simplify both sides until you find a familiar fact from calculus.

Solution: The product rule gives $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$. The $n-1$ form $\alpha \wedge \beta$ vanishes on both boundary components, so by Stokes' theorem,

$$0 = \int_{\partial M} \alpha \wedge \beta = \int_M d(\alpha \wedge \beta) = \int_M d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = \int_M d\alpha \wedge \beta + (-1)^p \int_M \alpha \wedge d\beta.$$

and therefore

$$\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta$$

This is integration by parts. In the special case described,

$$\int_M d\alpha \wedge \beta = \int_a^b f'(x)(g(x) - g(b))dx = \int_a^b f'(x)g(x)dx - \int_a^b f'(x)g(b)$$
$$= \int_a^b f'(x)g(x)dx - f(b)g(b) + f(a)g(b).$$

Similarly, $\int_M \alpha \wedge d\beta = \int_a^b f(x)g'(x)dx - f(a)g(b) + f(a)g(a)$, and together

$$\int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx$$

2. If M is a compact orientable manifold, show there is no smooth map $f : M \to \partial M$ with $f|_{\partial M} = Id$. Do we really need M to be compact? What if we only have ∂M compact?

   Hint: Integrate the volume form on ∂M in two ways: over ∂M and by pulling back to M.

Solution: Suppose such an f exists. Let ω be a volume form on ∂M. Then $\int_{\partial M} \omega > 0$. Since f is the identity on ∂M, $f^*\omega|_{\partial M} = \omega$, so $\int_{\partial M} f^*\omega > 0$. On the other hand, using Stokes' theorem, and the fact that $d\omega = 0$,

$$\int_{\partial M} f^*\omega = \int_M d(f^*\omega) = \int_M f^*(d\omega) = 0,$$

which is a contradiction.

3. Suppose M is a compact and orientable n-manifold with no boundary. Let $\theta \in \bigwedge^{n-1}(M)$ be any $n-1$ form. Show that $d\theta$ must vanish at some point of M.

Solution: In fact $d\theta$ must vanish at some point of each connected component of M. So, restricting to one component we may assume M is connected. Let ω be an orientation n-form on M. Then $d\theta = f\omega$ for some $f \in C^\infty(M)$. Suppose f is non-vanishing on M. Since M is connected, and by replacing θ with $-\theta$ if necessary, $f > 0$ on M. Then $\int_M d\theta = \int_M f\omega > 0$. But Stokes' theorem gives $\int_M d\theta = \int_{\partial M} \theta = 0$ since M has no boundary. This contradiction shows f must vanish somewhere on M, and therefore $d\theta$ vanishes somewhere on M.

4. Let (X, Y) be stereographic coordinates on S^2, with metric volume form

$$\mu = \frac{4}{(1 + X^2 + Y^2)^2} dX \wedge dY.$$

Find the divergence Div $\frac{\partial}{\partial X}$.

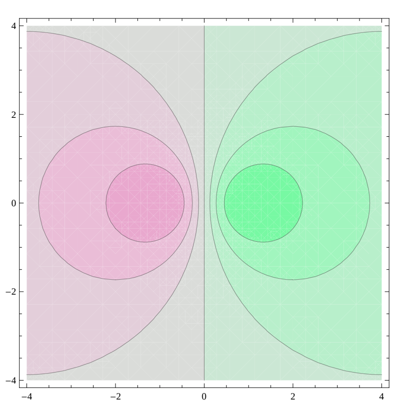
Where on the sphere is the divergence maximized and minimized? Make contour plots of Div $\frac{\partial}{\partial X}$ both in the planar (X, Y) coordinates and on the surface of the sphere. How does all this change for Div $\frac{\partial}{\partial Y}$?

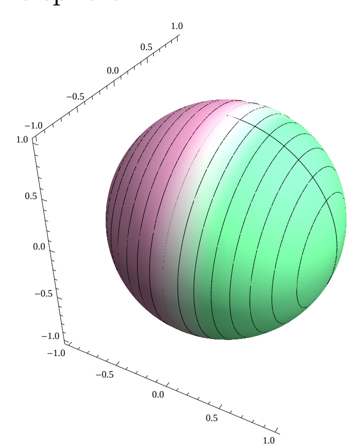
Solution: With $A = \frac{\partial}{\partial X}$, apply Cartan's formula:

$$\begin{aligned}\mathcal{L}_A\mu = d\iota_A\mu + \iota_A d\mu = d\iota_A \frac{4}{(1 + X^2 + Y^2)^2} dX \wedge dY = d\left(\frac{4}{(1 + X^2 + Y^2)^2} dy\right) \\ = \frac{-16X}{(1 + X^2 + Y^2)^3} dX \wedge dY = \frac{-4X}{1 + X^2 + Y^2}\mu\end{aligned}$$

So Div $\frac{\partial}{\partial X} = \frac{-4X}{1+X^2+Y^2}$. This has minimum -2 when $(X, Y) = (1, 0)$ and maximum 2 when $(X, Y) = (-1, 0)$. Changing to (x, y, z) coordinates, Div $\frac{\partial}{\partial X} = -2x$, and is minimized at $(1, 0, 0)$ and maximized at $(-1, 0, 0)$ on the unit sphere.

Plots in the (X, Y) plane and on the sphere:





With Y, Div $\frac{\partial}{\partial Y} = \frac{-4Y}{1+X^2+Y^2} = -2y$ and rotate both plots by $90°$ around the z axis.