Week 11 Exercises

1. For $X \in \mathfrak{X}(M)$, Cartan's formula relates Lie derivative with exterior and interior derivatives:

$$\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X$$

Prove this is true on $\bigwedge^1(M)$, i.e. prove

$$\mathcal{L}_X\theta = (\iota_X \circ d + d \circ \iota_X)\theta$$

for a one-form θ.

Solution: Let $X, Y \in \mathfrak{X}(M)$. The product rule for Lie derivatives gives:

$$\mathcal{L}_X(\theta(Y)) = \mathcal{L}_X\theta(Y) + \theta(\mathcal{L}_X(Y))$$

so that $\mathcal{L}_X\theta(Y) = \mathcal{L}_X(\theta(Y)) - \theta(\mathcal{L}_X(Y)) = X.\theta(Y) - \theta([X,Y])$. On the other hand,

$$\begin{aligned} ((\iota_X \circ d + d \circ \iota_X)\theta)(Y) &= (\iota_X d\theta)(Y) + d(\theta(X))(Y) \\ &= d\theta(X,Y) + Y.\theta(X) \\ (*) &= X.\theta(Y) - Y.\theta(X) - \theta([X,Y]) + Y.\theta(X) \\ &= X.\theta(Y) - \theta([X,Y]) \end{aligned}$$

where (*) used the invariant definition of d, which we showed in class in this 1-form case and which appears in its general form on Lee pg. 366.

- Lee Chapter 9 Problem 1

Solution: In spherical coordinates $(x,y,z) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$ for $\theta \in [0, 2\pi)$, $\phi \in (-\pi/2, \pi/2)$. Then

$$\begin{aligned} dx &= -\sin\theta\cos\phi\ d\theta - \cos\theta\sin\phi\ d\phi \\ dy &= \cos\theta\cos\phi\ d\theta - \sin\theta\sin\phi\ d\phi \\ dz &= \cos\phi\ d\phi \end{aligned}$$

So

$$\begin{aligned} \iota^*\omega &= \cos^2\theta\cos^3\phi d\theta \wedge d\phi - \sin^2\theta\cos^3\phi d\phi \wedge d\theta + \sin^2\phi\cos\phi d\theta \wedge d\phi \\ &= (\cos^3\phi + \sin^2\phi\cos\phi) d\theta \wedge d\phi = \cos\phi d\theta \wedge d\phi. \end{aligned}$$

Now

$$\int_{S^2} \iota^*\omega = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos\phi d\phi d\theta = \int_0^{2\pi} 2 d\theta = 4\pi.$$

- Lee Chapter 1 Exercise 1.127

Solution: For example, $[0,1] \times [0,1]$ is not a manifold with boundary, because of the corners.

To be more precise, this question should specify how the differentiable structure is placed on the product manifold. The usual method is, given charts (U, φ) on M and (V, ψ) on N, get a chart $(U \times V, \varphi \times \psi)$ on $M \times N$. This product chart will fail to be a homeomorphism into $\mathbb{R}^{m+n}_{\leq}$ on $\partial M \times \partial N$. On the other hand, it *is* a homeomorphsim into the quadrant $Q = \{(x_1, \ldots, x_{m+n}) | x_1 \leq 0, x_{m+1} \leq 0\}$, and Q itself is homeomorphic to the half space $\mathbb{R}^{m+n}_{\leq}$ by a continuous map which is not smooth. The upshot is that $M \times N$ *could* be made into a manifold with boundary, but not in any natural way.

- Don't write this one down, but think a little bit about Lee Exercise 1.128.

Solution: The definition of partition of unity goes through without a change. Existence of partitions of unity follow from the existence of a bump function on $\mathbb{R}^n$. Extending to the manifold with boundary case requires only the observation that a point x on the boundary of $\mathbb{R}^n_{\leq}$ also has a smooth bump function supported in an arbitrarily small neighborhood of x. In fact, you just take a bump function for x on $\mathbb{R}^n$ and restrict it to the half-space.

Generally, suppose X is a topological space and $\mathcal{C}$ is some algebra of functions from $X \to \mathbb{R}$. If there are bump functions in $\mathcal{C}$ then there is a partition of unity in $\mathcal{C}$. That is, if every $x \in X$ and U neighborhood of x has a function f with support in U and $f(x) \neq 0$, then there is a partition of unity consisting of functions in $\mathcal{C}$. The obvious examples are when X is a C^r-manifold and $\mathcal{C}$ are the C^r functions on M. The main nonexample is when M is a real (or complex) manifold and $\mathcal{C}$ is the real (or complex) analytic functions from M to $\mathbb{R}$ (or $\mathbb{C}$). Another good nonexample is when $X = \mathbb{R}^n$ and $\mathcal{C}$ is the polynomials.