Week 10 Exercises

1. For a (M, Ω) an oriented manifold, and $f : M \to M$ a diffeomorphism, say that f is orientation reversing if $f^*(\Omega) = \lambda\Omega$ for some $\lambda \in C^\infty(M)$ with $\lambda < 0$. If M is connected, show that every diffeomorphism $f : M \to M$ is either orientation preserving or orientation reversing. Give a counterexample when M is not connected.

Solution: Since $\bigwedge^n(T_pM)$ is one dimensional, $f^*(\Omega)_p = \lambda(p)\Omega_p$. Since f is a diffeomorphism, λ is non-vanishing, so $\lambda : M \to (-\infty, 0) \cup (0, \infty)$. M is connected, so either λ is always positive or λ is always negative, which correspond to orientation preserving and orientation reversing for f.

Let M be the disjoint union of two copies of S^2, and let f be the identity on one copy and the antipodal map on the other copy. Then f is neither orientation preserving or reversing. The function λ is $+1$ on one copy of S^2 and -1 on the other.

- (Boothby pg. 219 #4) Compute the natural volume form on S^2 in coordinates. Choose spherical coordinates or stereographic coordinates or both if you want.

Solution:

Spherical: Here $(x, y, z) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$ for $\theta \in [0, 2\pi), \phi \in (-\pi/2, \pi/2)$. Then

$$\begin{aligned} dx &= -\sin\theta\cos\phi\ d\theta - \cos\theta\sin\phi\ d\phi \\ dy &= \cos\theta\cos\phi\ d\theta - \sin\theta\sin\phi\ d\phi \\ dz &= \cos\phi\ d\phi \end{aligned}$$

Plugging these in to the metric $dx^2 + dy^2 + dz^2$ on $\mathbb{R}^3$ gives the metric $\cos^2\phi\ d\theta^2 + d\phi^2$ on S^2. The volume form is given by $\sqrt{g}d\theta \wedge d\phi$. In this case, the metric is the diagonal matrix $\begin{pmatrix} \cos^2\phi & 0 \\ 0 & 1 \end{pmatrix}$ which has determinant $g = \cos^2\phi$ and so the volume form is $\cos\phi d\theta \wedge d\phi$.

Stereographic: $(x, y, z) = \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1}\right)$ so

$$\begin{aligned} dx &= -\frac{2X^2 + 2Y^2 + 2}{(X^2 + Y^2 + 1)^2}dX - \frac{4XY}{(X^2 + Y^2 + 1)^2}dY \\ dy &= -\frac{4XY}{(X^2 + Y^2 + 1)^2}dX + \frac{2X^2 - 2Y^2 + 2}{(X^2 + Y^2 + 1)^2}dY \\ dz &= \frac{4X}{(X^2 + Y^2 + 1)^2}dX + \frac{4Y}{(X^2 + Y^2 + 1)^2}dY \end{aligned}$$

Now (after no small amount of work), $dx^2 + dy^2 + dz^2 = \frac{4}{(1+X^2+Y^2)^2}(dX^2 + dY^2)$. Then the volume form is

$$\sqrt{g}dX \wedge dY = \frac{4}{(1 + X^2 + Y^2)^2}dX \wedge dY$$

Note that in both cases, one could begin directly with the volume form $xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ on S^2.

- (Boothby pg. 226 #2) Practice computing exterior derivatives.

Solution:

(a) $d\phi = 0$, so ϕ is closed, and $\phi = d(xyz)$ so ϕ is exact.

(b) $d\phi = 2xy^2 dx \wedge dy + zdy \wedge dz$, so ϕ is not closed and therefore not exact.

(c) $d\theta = 0$, so θ is closed. If $\alpha = x^2y^2dy + yzdz$ then $\theta = d\alpha$, so θ is exact.