Week 1 Exercises

- Lee, Chapter 2, problem 19.

Solution: The integral curves satisfy $x' = x^2, y' = xy$, with initial point (x_0, y_0). When $x_0 = 0$, the integral curve is constant. Otherwise, solving the first equation gives $x = \frac{1}{x_0^{-1} - t} = \frac{x_0}{1 - x_0 t}$ (and note the second form works even when $x_0 = 0$). Plugging in to the equation for y', we need to solve $\frac{y'}{y} = \frac{x_0}{1 - x_0 t}$. This has the solution $y = \frac{y_0}{1 - x_0 t} = kx$ where $k = \frac{y_0}{x_0}$. The integral curve for (x_0, y_0) lies on the radial line through the origin and (x_0, y_0).

Written as a flow,

$$\Phi_t(x, y) = (1 - xt)^{-1}(x, y).$$

The flow is defined for $t \in (1/x, \infty)$ when $x < 0$, for $t \in (-\infty, 1/x)$ when $x > 0$, and for all t when $x = 0$.

- Lee, Chapter 2, problem 21.

Solution: The integral curves satisfy $x' = -y, y' = x$. These are circles around the origin. One easy way to see this is:

$$(x^2 + y^2)' = 2xx' + 2yy' = -2xy + 2yx = 0$$

which shows that $x^2 + y^2$ is constant and the curves lie on circles.

Another approach: Take the derivative of $x' = y$ to get $x'' = y' = -x$. This has the general solution $x(t) = a\cos(t) + b\sin(t)$. Now $a = x(0) = x_0$, and $b = x'(0) = -y_0$. Using $y = -x'$, we have:

$$x(t) = x_0 \cos(t) - y_0 \sin(t), \quad y(t) = x_0 \sin(t) + y_0 \cos(t).$$

Written as a flow:

$$\Phi_t(x, y) = (x\cos(t) - y\sin(t), x\sin(t) + y\cos(t)) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The flow is defined on all of $\mathbb{R}^2$ for all time.

For both 19 and 21, you should sketch the integral curves of X and find the maximal domain of definition of the flow, $\mathcal{D}_X$.

- Lee, Chapter 2, problem 22.

Solution: The vector field $\frac{\partial}{\partial \theta}$ is complete on $S^2 - (0, 0, 1)$. It is incomplete on $S^2 - p$ for any point p which is not the north or south pole.

- Lee, Chapter 2, problem 25. Hint: If $g(t)$ and $h(t)$ are curves in $GL(n, \mathbb{R})$, what is $\frac{d}{dt}(g(t)h(t))$? What is $\frac{d}{dt}(g^{-1}(t))$?

> **Solution:** Considering matrix entries, it's not hard to show $(gh)' = g'h + gh'$, and then that $(g^{-1})' = -g^{-1}g'g^{-1}$.
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> In this problem, we want $g' = g^2$, so carefully multiplying on the left and right by g^{-1} we get $g^{-1}g'g^{-1} = I$, and therefore $-(g^{-1})' = I$, so that $g^{-1} = -tI + g_0^{-1}$. The integral curve is given by $g(t) = (-tI + g_0^{-1})^{-1}$.
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> Note that for t small, $-tI + g_0^{-1}$ is invertible since g_0 is. However, $-tI + g_0^{-1}$ fails to be invertible when $\frac{1}{t}$ is an eigenvalue of g_0.

1. (The Homogeneity Lemma) For a connected smooth manifold M, let $p \neq q$ be any two points in M. Show that there is a diffeomorphism $\Phi : M \to M$ with $\Phi(p) = q$.

> **Solution:** The main argument runs as follows:
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> Let $c(t)$ be a embedded smooth curve $[0, 1] \to M$ joining p to q. Then $c'(t)$ is a vector field along the compact image of c. Extend c' to a compactly supported vector field X on all of M. Then the flow $\varphi^X(t, p)$ is complete, c is an integral curve for X, and so $\varphi^X(1, p) = q$. Then $\Phi(x) = \varphi^X(1, x)$ is the required diffeomorphism.
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> There are two tricky details. The first is producing the curve c, which takes a bit of an argument, but mainly comes down to the fact that M is locally Euclidean.
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> The second is extending the vector field. To do this, cover the image of the curve with pre-compact single-slice charts U_i, so that on U_i, the curve is given by $c(t) = (t, 0, 0, \ldots, 0)$. Since the image of the curve is compact, we can do this with finitely many such charts $U_1, \ldots, U_N$. On U_i, define $X_i = \frac{\partial}{\partial x_1}$, which extends $c'|_{U_i}$. Finally, let $U_0 = M - c([0, 1])$ be the complement of the curve, and define $X_0 = 0$. Choose a partition of unity ψ_i subordinate to the cover $U_0, U_1, \ldots, U_N$ and let $X(p) = \sum_i \psi_i(p)X_i(p)$. Then for $p = c(t)$ in the image of the curve, $\psi_0(p) = 0$ and
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> $$X(p) = \sum_{i=0}^{N} \psi_i(p)X_i(p) = \sum_{i=0}^{N} \psi_i(p)c'(t) = c'(t)\sum_{i=0}^{N} \psi_i(p) = c'(t).$$
>
> X is compactly supported since it's support is contained in the pre-compact set $U_1 \cup \cdots \cup U_N$.