Week 8-9 Exercises

1. Let X be the sterographic coordinate on S^1, given by $(x, y) \to \frac{x}{1-y} = X$. Write $\frac{\partial}{\partial X}$ in terms of the vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ on $\mathbb{R}^2$, and show $\frac{\partial}{\partial X}$ extends to a vector field on all of S^1.

2. Define a free and proper action of $\mathbb{Z}$ on $\mathbb{R}^3$ by $1 \cdot (x, y, z) = (x + 1, -y, -z)$. The quotient E has a vector bundle structure over S^1 with projection $\pi : E \to S^1$ by $\pi(x, y, z) = e^{2\pi i x}$. Show that E is a trivial $\mathbb{R}^2$ bundle over S^1, hence diffeomorphic to $S^1 \times \mathbb{R}^2$.

Solution: For $e^{i\theta} \in S^1$, define sections

$$X_1(e^{i\theta}) = (\frac{\theta}{2\pi}, \cos(\theta/2), -\sin(\theta/2)) \in E \tag{1}$$
$$X_2(e^{i\theta}) = (\frac{\theta}{2\pi}, \sin(\theta/2), \cos(\theta/2)) \in E \tag{2}$$

These are well defined since

$$X_1(e^{i(\theta+2\pi)}) = (\frac{\theta}{2\pi} + 1, \cos(\theta/2 + \pi), -\sin(\theta/2 + \pi)) \tag{3}$$
$$= (\frac{\theta}{2\pi} + 1, -\cos(\theta/2), \sin(\theta/2)) \tag{4}$$
$$= 1 \cdot (\frac{\theta}{2\pi}, \cos(\theta/2), -\sin(\theta/2)) = 1 \cdot X_1(e^{i\theta}) \tag{5}$$

so that different choices of θ map to the same point of E. The case for X_2 is similar. Finally, X_1 and X_2 are a frame for E since

$$\det \begin{pmatrix} X_1(e^{i\theta}) \\ X_2(e^{i\theta}) \end{pmatrix} = \det \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2)) \end{pmatrix} = 1 \neq 0.$$

3. Read over section 2.4, which shows how $T_{(p,q)}(M \times N) = T_pM \times T_pN$. Show that if M and N are parallelizable manifolds, then so is $M \times N$.

Solution: Let $X_1, \ldots, X_m$ be vector fields on M that give a frame for TM, and $Y_1, \ldots, Y_n$ be a frame for TN. Since $T_{(p,q)}(M \times N) = T_pM \times T_pN$, we get vector fields $(X_1, 0), \ldots, (X_m, 0)$ and $(0, Y_1), \ldots, (0, Y_n)$ on $M \times N$. These are clearly independent, and so form a frame for $M \times N$, so $M \times N$ is parallelizable.