Week 7 Exercises

1. With $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$, let $f : S^2 \to S^2$ by $f(\mathbf{x}) = -\mathbf{x}$ be the antipodal map. Let (X, Y) be stereographic projection coordinates from the north pole $(0, 0, 1)$. For $p \in S^2$, find $T_p f(\frac{\partial}{\partial X})$ and $T_p f(\frac{\partial}{\partial Y})$.

- Lee, Chapter 2, problem 2. The hint refers to Theorem 2.25, the Inverse Mapping Theorem for manifolds.

- Lee, Chapter 2, problem 12. The problem needs some clarification, because it's not clear what sort of structure the set N comes with. At least, you should find a bijection $f : TS^n \to N$. Better, you want f to be a smooth map into $\mathbb{R}^{2n+2}$. Even better, you could show f is an immersion, as defined in Chapter 3 (which amounts to showing that Tf has rank $2n$). Finally, you might also ask that f respect the vector space structures. For $x \in \mathbb{R}^{n+1}$, the set $V_x = \{y \in \mathbb{R}^{n+1} | (x, y) \in N\}$ is a vector space, and f should be a linear isomorphism from $T_p S^n$ to $V_{f(p)}$ for each $p \in S^n$.

  The main point of this problem was simply define a map f that has all these properties. Probably you should show it is smooth. Then, prove as much of the rest as you wish.

- Lee, Chapter 2, problem 20.