Week 6 Exercises

Let p be a point in an n-manifold M.

1. In the 'kinematic' tangent space, curves c_1, c_2 through p are equivalent if $(f \circ c_1)'(0) = (f \circ c_2)'(0)$ for all $f \in C^\infty(M)$. Show that c_1 is equivalent to c_2 if and only if $(f \circ c_1)'(0) = (f \circ c_2)'(0)$ for all $f \in C^\infty(U)$, where U is any neighborhood of p.

2. Given an element of the 'physical' tangent space $\mathbf{v} \in \mathbb{R}^n$ with chart (U, φ), define a curve $c(t) = \varphi^{-1}(\varphi(p) + t\mathbf{v})$. Show that this gives a well defined map from the 'physical' tangent space to the 'kinematic' tangent space.

3. On the 2-sphere $S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\}$, define $f(x, y, z) = z$. Stereographic projection from $(0, 0, 1)$ gives coordinates $(X, Y) = \frac{1}{1-z}(x, y)$. At $p = (x, y, z) \neq (0, 0, 1)$, the tangent vectors $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ are a basis for $T_p(S^2)$. Calculate $\frac{\partial}{\partial X} f|_p$ and $\frac{\partial}{\partial Y} f|_p$. Check that both approach 0 as $p \to (0, 0, 1)$.

4. Lee, Chapter 2, Problem 18. Show that the vector space of derivations at p on $C^r(M)$ is not isomorphic to $T_p(M)$, for non-smooth functions ($r < \infty$).