Math 641

# Week 5 Exercises

1. $\mathbb{Z}$ acts freely and properly on $\mathbb{R}^n - \{\mathbf{0}\}$ by $n \cdot \mathbf{v} = e^n \mathbf{v}$. When $n = 2$, show that the quotient manifold is diffeomorphic to the torus $\mathbb{T}^2$. Can you identify the quotient manifold for $n > 2$?

2. Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \big| |z_1|^2 + |z_2|^2 = 1\}$. For a given $\theta$, rotation by $\theta$ in the first coordinate is $(z_1, z_2) \to (e^{i\theta} z_1, z_2)$. Stereographic projection from $(1, 0)$ takes $(x + iy, z + iw)$ to $(Y, Z, W) = \frac{1}{1-x}(y, z, w)$. Show that rotation by $\theta$ (as above) maps the plane $Y = 0$ to the sphere with center $(\cot\theta, 0, 0)$ and radius $\csc\theta$.

   (This is basically a trigonometry exercise.)