Week 4 Exercises

1. Let M be the configuration space of two distinct points on the circle. Identify this with a well known 2-manifold.

2. Identify $\mathbb{R}P^1$ with a well known 1-manifold and give a diffeomorphism.

3. (Lee Exercise 1.46) Show that $\mathbb{C}P^1$ is diffeomorphic to S^2.

4. Suppose M is a smooth manifold and $f : M \to \mathbb{R}$ is continuous. Prove for any $\varepsilon > 0$ there is a C^∞ function $g : M \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in M$.

   Hint: Use the Stone-Weierstrauss theorem: If $K \subset \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}$ is continuous, then for all $\varepsilon > 0$ there is a polynomial g such that $|f(x) - g(x)| < \varepsilon$ for all $x \in K$.

5. Let $\mathbb{T}^2$ be the 2-torus, given as $\mathbb{T}^2 = \left\{(e^{i\theta}, e^{i\phi}) \in \mathbb{C}^2 | \theta, \phi \in \mathbb{R}\right\}$. For real numbers α, β, there is an action of $\mathbb{R}$ on $\mathbb{T}^2$ given by:

   $$t \cdot (e^{i\theta}, e^{i\phi}) = (e^{i(\theta + t\alpha)}, e^{i(\phi + t\beta)}).$$

   For which α, β is the quotient space $C = \mathbb{R} \backslash \mathbb{T}^2$ Hausdorff?
   (Bonus: When C is Hausdorff, what is C?)

6. (Lee Ch 1 Exercise 14) The definition of a Lens space.