Week 4 Exercises

1. Let M be the configuration space of two distinct points on the circle. Identify this with a well known 2-manifold.

2. Identify $\mathbb{R}P^1$ with a well known 1-manifold and give a diffeomorphism.

3. (Lee Exercise 1.46) Show that $\mathbb{C}P^1$ is diffeomorphic to S^2.

4. Suppose M is a smooth manifold and $f : M \to \mathbb{R}$ is continuous. Prove for any $\varepsilon > 0$ there is a C^∞ function $g : M \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in M$.

   Hint: Use the Stone-Weierstrauss theorem: If $K \subset \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}$ is continuous, then for all $\varepsilon > 0$ there is a polynomial g such that $|f(x) - g(x)| < \varepsilon$ for all $x \in K$.

Solution: Let $\{U_\alpha\}_{\alpha \in A}$ be a locally finite open cover of M so that each U_α has compact closure contained in a chart (V_α, ψ_α). To get such a cover, take a chart (V_p, ψ_p) for each $p \in M$, choose $B_p \subset V_p$ an open neighborhood of p with compact closure $\bar{B}_p \subset V_p$, and then apply paracompactness to the open cover $\{B_p\}_{p \in M}$.

Let $\{\varphi_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to $\{U_\alpha\}$.

Fix $\epsilon > 0$ and any $\alpha \in A$. Now $f \circ \psi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is continuous, so Stone-Weierstrauss gives a smooth function (polynomial, actually) $f_\alpha : V_\alpha \to \mathbb{R}$ so that $|f \circ \psi_\alpha^{-1}(x) - f_\alpha(x)| < \epsilon$ for all $x \in \psi(\bar{U}_\alpha)$.

Let $g_\alpha : M \to \mathbb{R}$ be $g_\alpha(p) = \varphi_\alpha(p) \cdot f_\alpha(\psi_\alpha(p))$ for $p \in V_\alpha$, and $g_\alpha(p) = 0$ otherwise. The function g_α is smooth, since it is smooth on V_α and any $p \in M - V_\alpha$ has a neighborhood on which g_α is identically zero. Also, the support of g_α is contained in U_α.

Define the smooth function $g(p) = \sum_{\alpha \in A} g_\alpha(p)$, which is a finite sum for any $p \in M$ by the local finiteness of $\{U_\alpha\}$.

Finally, for any p,

$$\begin{aligned}
|f(p) - g(p)| &= |f - \sum_\alpha g_\alpha(p)| \\
&= \left| \sum_\alpha \varphi_\alpha(p) f(p) - \sum_\alpha \varphi_\alpha(p) \cdot f_\alpha(\psi_\alpha(p)) \right| \\
&\leq \sum_\alpha \varphi_\alpha(p) \, |f(p) - f_\alpha(\psi_\alpha(p))| \\
&< \sum_\alpha \varphi_\alpha(p) \epsilon \\
&= \epsilon
\end{aligned}$$

5. Let $\mathbb{T}^2$ be the 2-torus, given as $\mathbb{T}^2 = \{(e^{i\theta}, e^{i\phi}) \in \mathbb{C}^2 | \theta, \phi \in \mathbb{R}\}$. For real numbers α, β, there is an action of $\mathbb{R}$ on $\mathbb{T}^2$ given by:

$$t \cdot (e^{i\theta}, e^{i\phi}) = (e^{i(\theta + t\alpha)}, e^{i(\phi + t\beta)}).$$

For which α, β is the quotient space $C = \mathbb{R}\backslash\mathbb{T}^2$ Hausdorff?
(Bonus: When C is Hausdorff, what is C?)

6. (Lee Ch 1 Exercise 14) The definition of a Lens space.