Week 3 Problem 5

Let M be the set of triangles in $\mathbb{R}^2$. Give an atlas on M to show it is a manifold.

We will define a chart given any triangle t. Let ρ_t be the length of the shortest side of t. Fix an ordering of the vertices of t and call them (t_1, t_2, t_3), with $t_i \in \mathbb{R}^2$. Let $U_i = B(t_i, \rho_t/4) \subset \mathbb{R}^2$ be the open ball of radius $\rho_t/4$ around t_i.

Let U_t be the set of triangles with one vertex in each of U_1, U_2, U_3. Then U_t is the domain of a chart $\varphi_t : U_t \to \mathbb{R}^6$ defined as follows: For any triangle $p \in U_t$, p has vertices $p_i = (x_i, y_i) \in U_i$, $i = 1, 2, 3$, and set $\varphi_t(p) = (p_1, p_2, p_3) = (x_1, y_1, x_2, y_2, x_3, y_3)$.

Note that the chart (U_t, φ_t) depends on t as well as the ordering of vertices of t, although that is not indicated in the notation. In fact, for each triangle t there are six (compatible) charts defined as above, depending on the choice of vertex order.

Now we must show these charts are all compatible, and so define an atlas.

Suppose s and t are triangles, with charts (U_s, φ_s) and (V_t, ψ_t) which overlap. Assume WLOG that $\rho_s \leq \rho_t$. Fix i. If $U_i \cap V_j \neq \emptyset$, then $d(s_i, t_j) < \rho_s/4 + \rho_t/4 < \rho_t/2$, which can only hold for at most one j (vertices of t are at least ρ_t apart). However, $U_i \cap V_j$ must be non-empty for some j because the charts to overlap. This means there is a permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ with $U_i \cap V_j \neq \emptyset$ if and only if $j = \sigma(i)$.

Let p be a triangle with $p \in U_s \cap V_t$. As before, p has one vertex in each of U_1, U_2, U_3, say $p_i \in U_i$. Also, p has one vertex in each of V_1, V_2, V_3, and in particular, $p_i \in V_{\sigma(i)}$.

The change of coordinate map is then $\psi_t \circ \varphi_s^{-1}(p_1, p_2, p_3) = (p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)})$, which is smooth.