Week 2 Exercises

1. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(r_1, r_2, r_3) = (c_0, c_1, c_2)$, where c_0, c_1, and c_2 are the coefficients of the monic polynomial $p(x) = (x - r_1)(x - r_2)(x - r_3) = x^3 + c_2 x^2 + c_1 x + c_0$. For which (r_1, r_2, r_3) does DF vanish? Away from these points, F has a smooth inverse and the roots of p are smooth functions of the coefficients of p.

   Challenge: Generalize to a degree n polynomial. Vandermonde determinants may be helpful here.

2. Consider a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as an element of $\mathbb{R}^4$. Then $\det : \mathbb{R}^4 \to \mathbb{R}$ by $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$. Compute $D \det|_I$, the derivative of det at the identity matrix. What is

$$D \det|_I \begin{pmatrix} a & b \\ c & d \end{pmatrix}?$$

3. For $b > 0$ and $c < 3b^2/4$, let S be the set of solutions to

$$x^4 + y^4 + z^4 - b(x^2 + y^2 + z^2) + c = 0.$$

   These are known as Goursat's surfaces. For which values of (b, c) is S a manifold? (At this point, to be a manifold we just want differentiable local coordinates)