Week 14 Exercises

If you want more practice with curves, Lee Chapter 4 problems 5 and 6 are good questions, although you may want to use software to help with the calculations in problem 6.

1. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is smooth, and a is a regular value of f. Show that $f^{-1}(a)$ is an orientable surface. (Part of Lee, Problem 4.12).

2. Lee Exercise 4.15: A connected, orientable hypersurface has exactly two unit normal fields.

3. Show that the connection $\overline{\nabla}$ respects the metric (see page 153 of Lee):

$$X.\langle Y, Z\rangle = \left\langle \overline{\nabla}_X Y, Z\right\rangle + \left\langle Y, \overline{\nabla}_X Z\right\rangle.$$

4. Lee, Example 4.63: The *tractrix* is the curve followed by a point $P = (x, y)$, starting at $(a, 0)$, when pulled at distance a by a point Q which starts at the origin and moves up the y-axis. The curve satisfies $\frac{dy}{dx} = -\frac{\sqrt{a^2-x^2}}{x}$. See Wikipedia for more information.

   The Beltrami sphere, or *psuedosphere*, is the horn-shaped surface created by rotating the tractrix around the y-axis. Find the shape operator on the psuedosphere, and show the surface has constant Gauss curvature $-\frac{1}{a^2}$.

5. Lee, Problem 4.13. Calculate the shape operator of a cylinder. What is the Gauss curvature?

If you want more practice, Problem 4.17 and Problem 4.19 have interesting surfaces. 4.19c is particularly worthwhile - calculating the curvature of an embedded torus.