Week 14 Exercises

If you want more practice with curves, Lee Chapter 4 problems 5 and 6 are good questions, although you may want to use software to help with the calculations in problem 6.

1. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is smooth, and a is a regular value of f. Show that $f^{-1}(a)$ is an orientable surface. (Part of Lee, Problem 4.12).

Solution: Let $M = f^{-1}(a)$, a manifold since a is a regular value. Also because a is a regular value, for any $p \in f^{-1}(a)$, the gradient $\nabla f(p) = (\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p)) \neq 0$.

Let $N = \frac{\nabla f}{||\nabla f||}$. Then N is a smooth unit vector field defined on all of M. Suppose $v \in T_p(M)$. Let c be a curve representing v, so $c(0) = p$ and $c'(0) = v$. Since $c(t) \in M$ for all t, we have $f(c(t)) = a$ for all t. Now

$$0 = \frac{\partial}{\partial t} f(c(t))\Big|_{t=0} = \nabla f(p) \cdot c'(0) = N(p) \cdot v,$$

which shows that N is normal to M at p.

2. Lee Exercise 4.15: A connected, orientable hypersurface has exactly two unit normal fields.

Solution: Let N be a unit normal field on the surface M, which exists by orientability. Let X be any unit normal field on M. The space of normal vectors to M at p is 1-dimensional (it is orthogonal to the $n-1$ dimensional tangent space T_pM), so we can write $X_p = \sigma(p)N_p$, for some continous function σ. Since both X_p and N_p are unit vectors, $\sigma(p) = \pm 1$. Since M is connected, either $\sigma \equiv 1$ or $\sigma \equiv -1$, and so $X = \pm N$.

3. Show that the connection $\overline{\nabla}$ respects the metric (see page 153 of Lee):

$$X.\langle Y, Z\rangle = \left\langle \overline{\nabla}_X Y, Z\right\rangle + \left\langle Y, \overline{\nabla}_X Z\right\rangle.$$

Solution:

$$X.\langle Y, Z\rangle = X.\sum_i Y_i Z_i = \sum_i X.(Y_i Z_i) = \sum_i (X.Y_i)Z_i + Y_i(X.Z_i) \tag{1}$$

$$= \sum_i (X.Y_i)Z_i + \sum_i Y_i(X.Z_i) = \left\langle \overline{\nabla}_X Y, Z\right\rangle + \left\langle Y, \overline{\nabla}_X Z\right\rangle. \tag{2}$$

4. Lee, Example 4.63: The *tractrix* is the curve followed by a point $P = (x, y)$, starting at $(a, 0)$, when pulled at distance a by a point Q which starts at the origin and moves up the y-axis. The curve satisfies $\frac{dy}{dx} = -\frac{\sqrt{a^2-x^2}}{x}$. See Wikipedia for more information.

The Beltrami sphere, or *psuedosphere*, is the horn-shaped surface created by rotating the tractrix around the y-axis. Find the shape operator on the psuedosphere, and show the surface has constant Gauss curvature $-\frac{1}{a^2}$.

Solution: Parameterize the surface as $R(x,\theta) = (x\cos\theta, x\sin\theta, y)$ where y is implicitly a function of x. The tangent vectors in the x and θ directions are

$$T_x := TR(\partial/\partial x) = (\cos\theta, \sin\theta, -\frac{\sqrt{a^2-x^2}}{x}); \quad T_\theta := TR(\partial/\partial\theta) = (-x\sin\theta, x\cos\theta, 0).$$

A normal field $\mathbf{n} = T_x \times T_\theta = (\sqrt{a^2-x^2}\cos\theta, \sqrt{a^2-x^2}\sin\theta, x)$. Since $||\mathbf{n}|| = a$, $N = \mathbf{n}/a$ is a unit normal field. For the shape operator,

$$S(T_x) = -\frac{\partial}{\partial x}N = -\frac{1}{a}(\frac{-x}{\sqrt{a^2-x^2}}\cos\theta, \frac{-x}{\sqrt{a^2-x^2}}\sin\theta, 1) = \frac{x}{a\sqrt{a^2-x^2}}T_x.$$

$$S(T_\theta) = -\frac{\partial}{\partial\theta}N = -\frac{1}{a}(-\sqrt{a^2-x^2}\sin\theta, \sqrt{a^2-x^2}\cos\theta, 0) = -\frac{\sqrt{a^2-x^2}}{ax}T_\theta$$

So the shape operator is given by the matrix

$$S = \begin{pmatrix} \frac{x}{a\sqrt{a^2-x^2}} & 0 \\ 0 & -\frac{\sqrt{a^2-x^2}}{ax} \end{pmatrix}$$

which has determinant $K = \det(S) = -\frac{1}{a^2}$.

5. Lee, Problem 4.13. Calculate the shape operator of a cylinder. What is the Gauss curvature?

Solution: Parameterize the surface by $(z,\theta) \to (r\cos\theta, r\sin\theta, z)$. Then $\frac{\partial}{\partial\theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$. The length of $\frac{\partial}{\partial z} \times \frac{\partial}{\partial\theta}$ is r, so a unit normal vector field is given by

$$N = \frac{1}{r}\frac{\partial}{\partial z} \times \frac{\partial}{\partial\theta} = -\cos\theta\frac{\partial}{\partial x} - \sin\theta\frac{\partial}{\partial y}.$$

Since $\frac{\partial}{\partial z}N = 0$, $S(\frac{\partial}{\partial z}) = 0$, and

$$S(\frac{\partial}{\partial\theta}) = -\overline{\nabla}_{\frac{\partial}{\partial\theta}}N = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial\theta}$$

We find that in the (orthonormal) basis $\frac{\partial}{\partial z}, \frac{\partial}{\partial\theta}$, the shape operator is given by the matrix $S = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$ and the Gauss curvature $K = \det S = 0$.

If you want more practice, Problem 4.17 and Problem 4.19 have interesting surfaces. 4.19c is particularly worthwhile - calculating the curvature of an embedded torus.