Week 12 Exercises

1. Lee Exercise 3.10: Steiner's Roman surface as an immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$ and an embedding into $\mathbb{R}^4$. Beware that in the definitions of f and g, Lee is assuming $(x, y, z) \in S^2$, i.e. $x^2 + y^2 + z^2 = 1$.

Solution:

(a) To get a map $\mathbb{R}P^2 \to \mathbb{R}^3$, define $f([x : y : z]) = (x^2 + y^2 + z^2)^{-1}(yz, xz, xy)$. $\mathbb{R}P^2$ is covered by three charts, corresponding to $x \neq 0$, $y \neq 0$, and $z \neq 0$.

When $x \neq 0$, we have coordinates $(y, z) \to [1 : y : z]$, and

$$f([1 : y : z]) = (1 + y^2 + z^2)^{-1}(yz, z, y).$$

Compute

$$Tf = (1 + y^2 + z^2)^{-2} \begin{pmatrix} -y^2 z + z^3 + z & y^3 - yz^2 + y \\ -2yz & y^2 - z^2 + 1 \\ -y^2 + z^2 + 1 & -2yz \end{pmatrix}$$

The determinant of the bottom two rows is (after a bit of simplification) $\frac{y^2+z^2-1}{(y^2+z^2+1)^3}$, so Tf has rank 2 unless $y^2 + z^2 - 1 = 0$. Assuming this, $y^2 + z^2 = 1$ and Tf simplifies to

$$Tf = \frac{1}{2} \begin{pmatrix} z^3 & y^3 \\ -yz & y^2 \\ z^2 & -yz \end{pmatrix}$$

The determinant of the top two rows of this matrix is $z^3 y^2 + y^4 z = zy^2(z^2 + y^2) = zy^2$, so Tf has rank 2 unless $y^2 + z^2 = 1$ and $zy^2 = 0$. This gives $(y, z) = (\pm 1, 0)$ and $(y, z) = (0, \pm 1)$ as points where Tf may have rank less than 2, and it is easy to see that at these four points Tf actually has rank 1.

In this chart, where $x = 1$, we have four points where f fails to be an immersion: $[1 : \pm 1 : 0]$ and $[1 : 0 : \pm 1]$. Similarly, when $y = 1$ we find f fails to be an immersion at the four points $[\pm 1 : 1 : 0]$ and $[0 : 1 : \pm 1]$, and when $z = 1$ at the four points $[\pm 1 : 0 : 1]$, $[0 : \pm 1 : 1]$. These points are equal in pairs, and so f is an immersion except at the six points

$$[1 : 0 : 1], [1 : 1 : 0], [0 : 1 : 1], [1 : 0 : -1], [-1 : 1 : 0], [0 : -1 : 1]$$

(b) Write $g([x, y, z]) = (f([x : y : z]), h([x : y : z]))$, where $h([x, y, z]) = (x^2 + y^2 + z^2)^{-1}(ax^2 + by^2 + cz^2)$. The problem specified $a = 1, b = 2, c = 3$ but we treat the general case to avoid differences among the three charts. Since Tf has rank two everywhere but the six exceptional points, Tg will be rank two except possibly at those points. In the chart $[1 : y : z]$, compute

$$\frac{\partial h}{\partial y} = \frac{2y(b - a + (b - c)z^2)}{(1 + y^2 + z^2)^2}, \quad \frac{\partial h}{\partial z} = \frac{2z(c - a + (c - b)y^2)}{(1 + y^2 + z^2)^2}.$$

Then

$$Tg_{[1:0:\pm 1]} = \frac{1}{2} \begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \pm(c - a) \end{pmatrix}, \quad Tg_{[1:\pm 1:0]} = \frac{1}{2} \begin{pmatrix} 0 & \pm 1 \\ 0 & 1 \\ 0 & 0 \\ \pm(b - a) & 0 \end{pmatrix}$$

These have rank 2 as long as a, b, c are distinct. By symmetry among the charts, Tg has rank 2 at all six exceptional points, and therefore g is an immersion.

To show g is injective, restrict to S^2, where $x^2 + y^2 + z^2 = 1$, and $g = (f_1, f_2, f_3, h) = (yz, xz, xy, ax^2 + by^2 + cz^2)$.

If none of f_1, f_2, f_3 are zero, then x, y, and z are uniquely determined (up to sign) by g, since $x^2 = \frac{(xy)(xz)}{(yz)}$ and similarly for y^2, z^2. Changing the sign of any one or two of x, y, z changes the value of g, so when f_1, f_2, f_3 are all nonzero, $[x : y : z] = [-x : -y : -z]$ is determined by $g([x : y : z])$.

It is not possible for just one of f_1, f_2, f_3 to vanish.

Suppose exactly two of f_1, f_2, f_3 vanish, say $f_3 \neq 0$. Then $x = 0$, and y, z are given by solving $yz = f_3$ and $y^2 + z^2 = 1$. This is an intersection of a hyperbola and the unit circle, and it has four solutions of the form $\pm(y, z), \pm(z, y)$. Then the possible preimages of $(0, 0, f_3, h)$ are $[0 : y : z]$ and $[0 : z : y]$, and the value of h distinguishes these.

Finally, if $f_1 = f_2 = f_3 = 0$, then two of x, y, z must vanish and the third must equal ± 1. Then h will be a, b, or c depending on which of x, y, z is nonzero.

Since g is an injective immersion, and $\mathbb{R}P^2$ is compact, g is an embedding.

2. When do $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^\ell$ intersect transversally in $\mathbb{R}^n$?

Solution: The tangent space to $\mathbb{R}^k \times \{0\}$ has basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$. The tangent space to $\{0\} \times \mathbb{R}^\ell$ has basis $\frac{\partial}{\partial x_{n-\ell+1}}, \ldots, \frac{\partial}{\partial x_n}$. Together, these span the tangent space to $\mathbb{R}^n$ as long as $n - \ell + 1 \leq k + 1$, or $k + \ell \geq n$. So $\mathbb{R}^k \times \{0\} \pitchfork \{0\} \times \mathbb{R}^\ell$ if and only if $k + \ell \geq n$.

3. Let V be a vector space, and let Δ be the diagonal of $V \times V$. For a linear map $A : V \to V$, let $\Gamma = \{(v, Av) | v \in V\}$ be the graph of A. Show that Γ intersects Δ transversally if and only if 1 is not an eigenvalue of A.

Solution: Identify $T\Delta$ with Δ and $T\Gamma$ with Γ (since these are vector spaces). Then $\Delta \pitchfork \Gamma$ if and only if $\Delta + \Gamma = V \times V$. If $\dim V = n$, Δ is an n-dimensional subspace of $V \times V$, and so is Γ. Then $\Delta + \Gamma = V \times V$ if and only if $\Delta \cap \Gamma = \{0\}$. Finally, $0 \neq v \in \Delta \cap \Gamma$, iff $(v, v) = (v, Av)$, iff $Av = v$, iff 1 is an eigenvalue of A.

4. Suppose M_1 and M_2 are regular submanifolds of N which intersect transversally. Show $M_1 \cap M_2$ is a submanifold of N. What is its dimension?

Solution: Let ι be the inclusion map $M_1 \to N$. Then $\iota \pitchfork M_2$, so $\iota^{-1}M_2$ is a submanifold of M_1. Since $\iota^{-1}(M_2) = M_1 \cap M_2$, $M_1 \cap M_2$ is a submanifold of M_1 and therefore of N.

If M_i has codimension k_i, then $M_1 \cap M_2$ has codimension $k_1 + k_2$ in N.

5. Lee Exercise 2.29: Transversality of composed maps. There is a typo in the problem statement, which should read: $f \pitchfork g^{-1}(W)$ if and only if $(g \circ f) \pitchfork W$.

Solution: I'll use better names for the manifolds involved. We have $f : X \to Y$, $g : Y \to Z$, and W a regular submanifold of Z. We know $g(Y) \pitchfork W$, and want to show $f \pitchfork g^{-1}(W) \iff (g \circ f) \pitchfork W$.

$\implies$: Let $z \in (g \circ f)(X) \cap W$, let $x \in X$ with $g(f(x)) = z$, and $y = f(x)$.

Fix $\mathbf{c} \in T_z Z$. Since $W \pitchfork g(Y)$, we can write write $\mathbf{c} = \mathbf{w} + Tg(\mathbf{v})$ for $\mathbf{w} \in T_z W$ and $\mathbf{v} \in T_y Y$. Since $f \pitchfork g^{-1}(W)$, we can write $\mathbf{v} = \mathbf{v}' + Tf(\mathbf{u})$ with $\mathbf{v}' \in T_y(g^{-1}(W))$ and $\mathbf{u} \in T_x X$. Let $\mathbf{w}' = Tg(\mathbf{v}') \in T_z W$. Finally

$$T(g \circ f)(\mathbf{u}) + \mathbf{w}' + \mathbf{w} = Tg(Tf(\mathbf{u}) + \mathbf{v}') + \mathbf{w} = Tg(\mathbf{v}) + \mathbf{w} = \mathbf{c}.$$

This shows that $T_z Z = T(g \circ f)(T_x X) + T_z W$ and therefore $(g \circ f) \pitchfork W$.

$\impliedby$: Let $y \in f(X) \cap g^{-1}(W)$, let $x \in X$ with $f(x) = y$, and $z = g(y)$.

Fix $\mathbf{b} \in T_y Y$, and let $\mathbf{c} = Tg(\mathbf{b})$. Since $(g \circ f) \pitchfork W$, we can write $\mathbf{c} = T(g \circ f)(\mathbf{u}) + \mathbf{w}$, with $\mathbf{u} \in T_x X$ and $\mathbf{w} \in T_z W$. Let $\mathbf{v} = \mathbf{b} - Tf(\mathbf{u})$. Then

$$Tg(\mathbf{v}) = Tg(\mathbf{b}) - Tg(Tf(\mathbf{u})) = \mathbf{c} - (\mathbf{c} - \mathbf{w}) = \mathbf{w}.$$

Since $g(Y) \pitchfork W$, Theorem 2.47 in Lee says that $T_y(g^{-1}(W)) = (Tg)^{-1}(T_z W)$, so $\mathbf{v} \in T_y(g^{-1}(W))$. Since $\mathbf{b} = \mathbf{v} + Tf(\mathbf{u})$, we have shown that $T_y Y = T_y(g^{-1}(W)) + Tf(T_x X)$, and so $f \pitchfork g^{-1}(W)$.

Note that (for me, anyway) everything in this proof was routine, since at each moment there is really only one thing you can possibly do. However, in the second part, the fact that $Tg(\mathbf{v}) \in TW \implies \mathbf{v} \in T(g^{-1}(W))$ stumped me for over an hour, until I realized it was the content of Theorem 2.47. That's why it's a Theorem!

Here's another way to do the $\implies$ part which avoids working with individual vectors: With x, y, z as above, $f \pitchfork g^{-1}(W))$ implies

$$T_y Y = Tf(T_x X) + T_y(g^{-1}(W)).$$

Apply Tg to both sides to get:

$$Tg(T_y Y) = T(g \circ f)(T_x X) + Tg(T_y(g^{-1}(W))).$$

Addd $T_z W$ to both sides:

$$Tg(T_y Y) + T_z W = T(g \circ f)(T_x X) + Tg(T_y(g^{-1}(W))) + T_z W.$$

Now $g \pitchfork W$, so the left hand side is $T_z Z$. Also $Tg(T_y(g^{-1}(W))) \subset T_z W$, so:

$$T_z Z = T(g \circ f)(T_x X) + T_z W$$

and therefore $(g \circ f) \pitchfork W$.