Week 11 Exercises

1. Lee Exercise 3.6: Show that every injective immersion of a compact manifold is an embedding.

Solution: Given $f : M \to N$, we need to show f is a homeomorphism from M to $f(M)$ with the subspace topology. This follows if f^{-1} is continuous.

So, suppose $\{y_n\}_{n=1}^{\infty}$ is a sequence with $y_n \in f(M)$ and $y_n \to y \in f(M)$. Let $x_n = f^{-1}(y_n)$ and $x = f^{-1}(y)$. We need to show $x_n \to x$.

Suppose not, so x_n does not converge to x. Then by passing to a subsequence, there is an open neighborhood U of x with $x_n \notin U$ for all n. Then $M - U$ is compact, so, again passing to a subsequence, there is $x' \neq x$ with $x_n \to x'$. But f is continuous, so $f(x_n) \to f(x')$, so $y_n \to f(x')$, so $f(x') = y$. This contradicts injectivity of f, since $f(x) = y$.

Note that this fact doesn't require that f be an immersion, just an injective continuous map.

2. Show that any two loops in $\mathbb{R}^3$ can be nudged away from each other. More precisely:

   Given smooth maps $f : S^1 \to \mathbb{R}^3$ and $g : S^1 \to \mathbb{R}^3$, show that for any $\delta > 0$ there is a vector $\mathbf{v} \in \mathbb{R}^3$ with $||\mathbf{v}|| < \delta$ so that the sets $\{f(x)\}_{x \in S^1}$ and $\{g(x) + \mathbf{v}\}_{x \in S^1}$ are disjoint.

Solution: On the torus, define $F : \mathbb{T}^2 \to \mathbb{R}^3$ by $F(x, y) = f(x) - g(y)$. F is smooth, so by Sard's Theorem, the set of critical values of F is measure 0 in $\mathbb{R}^3$. In particular, there is a regular value $\mathbf{v}$ with $||\mathbf{v}|| < \delta$. However, $\dim \mathbb{T}^2 < \dim \mathbb{R}^3$, so a regular value must be a point not in the image of F. That is, $F(x, y) \neq \mathbf{v}$ for any $x, y \in S^1$. So $f(x) - g(y) \neq \mathbf{v}$, or $f(x) \neq g(y) + \mathbf{v}$ for any $(x, y) \in \mathbb{T}^2$. This shows the sets $\{f(x)\}_{x \in S^1}$ and $\{g(x) + \mathbf{v}\}_{x \in S^1}$ are disjoint.

3. Let M be any (paracompact) manifold which is not second countable. For example, M could be an uncountable disjoint union of circles. Show there is no embedding $M \to \mathbb{R}^n$ for any n, so the Whitney embedding theorem fails.

Solution: Suppose $f : M \to \mathbb{R}^n$ is an embedding. Let $\{U_i\}_{i=1}^{\infty}$ be a countable basis for the topology of $\mathbb{R}^n$. Then $\{U_i \cap f(M)\}_{i=1}^{\infty}$ is a countable basis for the toplogy of $f(M)$, which is homeomorphic to M. But M has no countable basis, a contradiction.