Week 10 Exercises

1. Lee Exercise 1.118. Given smooth $f : \mathbb{R}^n \to \mathbb{R}^m$, show the graph of f is a regular submanifold of $\mathbb{R}^m \times \mathbb{R}^n$.

2. Lee Exercise 1.119. The main point of this exercise is to show that every regular submanifold M of $\mathbb{R}^n$ is locally the graph of a function.

3. For a real number a, define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^3 - 3ax - y^2$. Find all values of b so that $f^{-1}(b)$ is a manifold. Graph $f^{-1}(b)$ for a variety of a and b, including the critical values of b. These manifolds are called elliptic curves.

4. Lee Ch 3 Problem 28. For a homogeneous polynomial p of n variables, show $p^{-1}(c)$ is a submanifold of $\mathbb{R}^n$ for all $c \neq 0$.

5. Lee Ch 3 Problem 3. Show if M is compact and N is connected, then a submersion $f : M \to N$ must be surjective.

6. Define the map $H : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$H(x, y, z, w) = (2(xy + zw), 2(xw - yz), x^2 + z^2 - y^2 - w^2).$$

   Check that restricting H to $S^3 \subset \mathbb{R}^4$ defines a map from S^3 to S^2, the Hopf map. Show that the Hopf map is a submersion. What are the fibers of the Hopf map (i.e. what manifold is $f^{-1}(q)$ for $q \in S^2$)?