Week 10 Exercises

1. Lee Exercise 1.118. Given smooth $f : \mathbb{R}^n \to \mathbb{R}^m$, show the graph of f is a regular submanifold of $\mathbb{R}^m \times \mathbb{R}^n$.

Solution: Define $\varphi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ by $\varphi(x, y) = (x, y - f(x))$. Clearly, $\varphi(x, y) = (x, 0)$ if and only if (x, y) is in the graph of f, i.e. $f(x) = y$. Next, if $\varphi(x', y') = \varphi(x, y)$ then $x = x'$ and so $y - f(x) = y' - f(x)$, and so $y = y'$, which shows φ is injective. Finally,

$$D\varphi = \begin{pmatrix} I & 0 \\ -Df & I \end{pmatrix}$$

which is nonsingular, so φ is a diffeomorphism. Thus φ is a single-slice chart for the graph of f, making graph f a regular submanifold of $\mathbb{R}^m \times \mathbb{R}^n$.

2. Lee Exercise 1.119. The main point of this exercise is to show that every regular submanifold M of $\mathbb{R}^n$ is locally the graph of a function.

Solution: Note this problem requires the Implicit Mapping Theorem. See Hebda's notes or Appendix C. It's not hard to prove, given the Inverse Function Theorem.

Call the standard coordinates on $\mathbb{R}^n$ $(u_1, \ldots, u_n)$. For $p \in M$, choose single slice coordinates $x_1, \ldots, x_n$ on a neighborhood U of p, so that $M \cap U = \{(x_1, \ldots, x_k, 0, \ldots, 0)\}$.

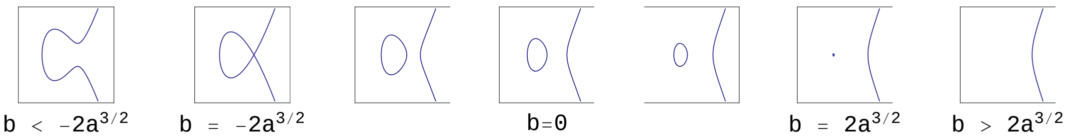
Consider each x coordinate as a function $x_i = x_i(u_1, \ldots, u_n)$. The matrix $\left(\frac{\partial x_i}{\partial u_j}\right)_{i,j=1\ldots n}$ is invertible (it's the derivative of the change of coordinates function), so the bottom $n-k$ rows form a rank n matrix $\left(\frac{\partial x_i}{\partial u_j}\right)_{i=k+1\ldots n, j=1\ldots n}$. This matrix has $n-k$ independent columns, and by renumbering the u_i we may assume that the $n-k \times n-k$ matrix $\left(\frac{\partial x_i}{\partial u_j}\right)_{i=k+1\ldots n, j=k+1\ldots n}$ is invertible. The coordinate plane P we're looking for has now been chosen as the plane spanned by $u_1, \ldots, u_k$, after the renumbering.

Define $f : U \to \mathbb{R}^{n-k}$ by $f(u_1, \ldots, u_n) = (x_{n-k+1}, \ldots, x_n)$. Then $M \cap U$ is the zero set of f, i.e. $M \cap U = f^{-1}(0)$.

By the Implicit Mapping Theorem, there is a smooth function $g : \mathbb{R}^k \to \mathbb{R}^{n-k}$ so that $f(u_1, \ldots, u_k, g(u_1, \ldots, u_k)) = 0$ if and only if $(u_{k+1}, \ldots, u_n) = g(u_1, \ldots, u_k)$. Then $(u_1, \ldots, u_k) \to (u_1, \ldots, u_k, g(u_1, \ldots, u_l))$ gives a smooth parameterization of $M \cap U$, showing that M is locally the graph of a function over the coordinate plane P, and so projection to P gives a coordinate chart on M.

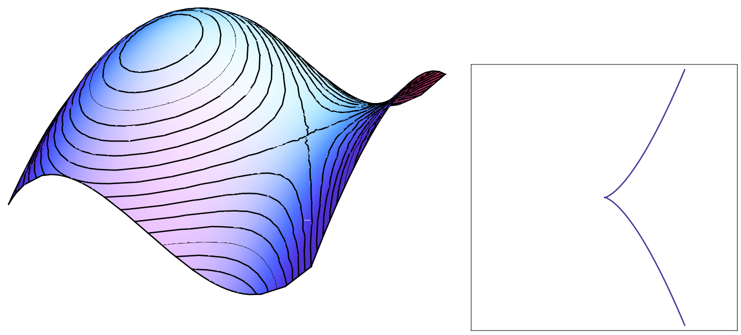
3. For a real number a, define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^3 - 3ax - y^2$. Find all values of b so that $f^{-1}(b)$ is a manifold. Graph $f^{-1}(b)$ for a variety of a and b, including the critical values of b. These manifolds are called elliptic curves.

Solution: Df either has rank 1 or is identically zero, and if it has rank 1, then $f^{-1}(b)$ is a manifold. Solve $Df = \begin{pmatrix} 3x^2 - 3a \\ -2y \end{pmatrix} = 0$. First, $y = 0$, and we must have $a \geq 0$ and then $x = \pm\sqrt{a}$. Then $f(\pm\sqrt{a}, 0) = \pm a^{3/2} \mp 3a^{3/2} = \pm 2a^{3/2}$. So the singular values for a given $a > 0$ are $\pm 2a^{3/2}$.



At $b = -2a^{3/2}$, the surface $f(x, y)$ has a saddle point, causing a crossing, and at $b = 2a^{3/2}$ there is a local maximum, causing an isolated point.

When $a = 0$, there is a single critical point $b = 0$ which is a cusp.



4. Lee Ch 3 Problem 28. For a homogeneous polynomial p of n variables, show $p^{-1}(c)$ is a submanifold of $\mathbb{R}^n$ for all $c \neq 0$.

Solution: For $\mathbf{x} \in p^{-1}(c)$, define a curve $\sigma(t) = (1+t)\mathbf{x}$. Then $p \circ \sigma : \mathbb{R} \to \mathbb{R}$. I will show $p \circ \sigma$ has rank 1 at $t = 0$ and therefore p has rank 1 at $\mathbf{x}$.

$$T_0(p \circ \sigma) = \frac{d}{dt} p(\sigma(t))\big|_{t=0} = \frac{d}{dt}(1+t)^m p(\mathbf{x})\big|_{t=0} = mc \neq 0.$$

Since p has rank 1 at $\mathbf{x}$ for all $x \in p^{-1}(c)$, c is a regular value for p and therefore $p^{-1}(c)$ is a manifold.

5. Lee Ch 3 Problem 3. Show if M is compact and N is connected, then a submersion $f : M \to N$ must be surjective.

Solution: We'll show $f(M)$ is both open and closed in N. First, if $y \in f(M)$ then $y = f(x)$ for some x. In local coordinates near x and y, f has the form $f(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$, so an open set around x maps to an open set around y, and $f(M)$ is open. Now suppose there is a sequence $y_1, y_2, \ldots$ converging to y in N, with $y_i = f(x_i)$. Since M is compact, by passing to a subsequence $y_i \to x \in M$. Since f is continuous, $f(y_i) \to f(x)$, so $y = f(x)$ and $f(M)$ is closed.

Since N is connected, $f(M) = N$ and f is surjective.

6. Define the map $H : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$H(x, y, z, w) = (2(xy + zw), 2(xw - yz), x^2 + z^2 - y^2 - w^2).$$

Check that restricting H to $S^3 \subset \mathbb{R}^4$ defines a map from S^3 to S^2, the Hopf map. Show that the Hopf map is a submersion. What are the fibers of the Hopf map (i.e. what manifold is $f^{-1}(q)$ for $q \in S^2$)?