

Aug. 27

1. Sketch the stereographic projection of a cube inscribed in the unit sphere. Make two sketches: One where the cube is aligned with the coordinate planes (so corners are at $(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}})$), and another where the cube has two corners at the poles $(0, 0, \pm 1)$
2. Bernhardus Varenius was a German geographer who died in 1650 at the age of 28, just after publishing *Geographia Generalis*, in which he describes a method for constructing the graticule (meridians and parallels) in a polar stereographic projection. The instructions (and a brief intro) are reproduced on the next page.

Make sense of these instructions and perform the construction.

3. Variants of stereographic projection. The standard stereographic projection is from the ‘North Pole’, the point $(0, 0, 1)$, to the x, y plane given by $z = 0$. Another reasonable projections is from the ‘South Pole’, or $(0, 0, -1)$ to the $z = 0$ plane, and a third is from the North Pole to the plane $z = -1$ which is tangent to the South Pole.

Find formulas for these projections.

Compute the change of coordinates between all three variants.

Aug. 29

1. For various constants c , the function $f(x, y, z) = x^2 + y^2 - z^2$ defines a surface by the equation $f(x, y, z) = c$. Find a parameterization of these surfaces analogous to the spherical coordinate parameterization of the 2-sphere.
2. For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, show that if f is differentiable at \mathbf{a} then f is continuous at \mathbf{a} .
3. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies $\|f(\mathbf{x})\| \leq \|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that f is differentiable at $\mathbf{0}$.

Aug. 31

1. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear. For $\mathbf{a} \in \mathbb{R}^n$, what is $Df(\mathbf{a})$?
2. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable, and has an inverse function $f^{-1} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ which is also differentiable. For $\mathbf{a} \in \mathbb{R}^n$, what is $Df^{-1}(\mathbf{a})$?
3. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-x^{-2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that the k^{th} derivative $f^{(k)}(0) = 0$ for all k , and f is smooth. (Hint: L'Hopital's rule.)

Let

$$g(x) = \begin{cases} f(x+1)f(x-1) & x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

Show that g is a smooth function which is positive on $(-1, 1)$ and 0 elsewhere.

Given $\varepsilon > 0$, define a smooth function h which is 0 for $x \leq 0$ and 1 for $x \geq \varepsilon$. (Hint: integrate g).

Varenius, 1650, reproduced from Snyder, *Flattening the Earth*, 1993, pg 21.

On the polar aspect, the meridians and parallels are the straight lines (intersecting at the central pole) and the concentric circles, respectively, that appear on all polar azimuthals, but the spacing of the circles gradually increases away from the pole (fig. 1.14). The radii of these circles are proportional to the trigonometric tangent of half the colatitude (90° minus the latitude, or the angular distance of the latitude from the pole). In 1650, without displaying a diagram, Varenius described the geometric construction of the polar aspect, named only as the "first easy Mode, the Eye being placed in the Axis," as follows, utilizing the perspective nature of the stereographic projection:

In any Plain or paper let the middle *point* P, be taken for the Pole, and from that as from a *Center*, let the great or small *Periphery* [circle] be drawn (as we desire to have our Maps great or small) which we shall have for the *Æquator*. These two may be taken at pleasure, but the other *points* and *Peripheries* shall be found from them. Let the *Æquator* be divided into 360 *deg.* and *streight lines* being drawn through the *Center* and the beginning of every *deg.* these shall be the Meridians, from which that which is drawn at the beginning of the first *degree* from these 360, shall be taken for the first, so the rest of the *lines* shall shew the rest of the Meridians and *Longitudes* of the Earth from the first Meridian. Now the Parallels of *Latitude* must be described. There are four *Quadrants*, or quarters of the *Æquator*, the first 0, 90: the second 90, 108 [*sic*; should be 180]: the third 180, 270: and the fourth 270, 0. Let those be noted for the more easy appellation with the letters AB, BC, CD, DA, and let one be taken from these, for *Example*, BC, from every one of whose *degrees* as also from the 20 [*sic*; should be 23] *deg.* 30 *min.* and the 66 *deg.* 30 *min.* let occult *streight lines* be drawn to the *point* D, (the term of the *Diameter* BD) or let the *Rule* be only applied to D, and brought round through every *degree* of the *Quadrant* BC: and let the 23 *deg.* 30 *min.* and the 66 *deg.* 30 *min.* in which these *streight lines* cut the *Semidiameter* PC, be noted, and from P as from a *Center*, and the *Peripheries* be described through every *point* taken in PC. These *Peripheries* shall be the Parallels of the *Lattitudes* unto which in the first, and opposite Meridian, *viz.* AP, and CP, the numbers may be ascribed from the *Æquator* towards P, *to wit*, 1, 2, 3, 4, even to 90. (1650; 1693 translation, 319, 320)