1. Let $f : \mathbb{R} \to \mathbb{R}$ be any smooth function, and define $X = f(y)\frac{\partial}{\partial x}$ a vector field on $\mathbb{R}^2$. Find the flow for $X$.

> **Solution:** Need $\varphi_t(x, y) = (x(t), y(t))$ with $x'(t) = f(y(t))$, $y'(t) = 0$, $x(0) = x_0$ and $y(0) = y_0$. This has solution $y(t) = y_0$ and $x(t) = x_0 + f(y_0)t$. So $\varphi_t(x, y) = (x + f(y)t, y)$.

2. Let $f : M \to N$ and $g : L \to M$ be smooth maps of manifolds. For a tensor $\tau \in \mathcal{T}^r(N)$, show that $(f \circ g)^*\tau = g^*(f^*\tau)$.

> **Solution:** Let $X_1, \dots, X_r$ be vectors in $T_p(L)$. Then
>
> $$\begin{aligned} (f \circ g)^*\tau(X_1, \dots, X_n) &= \tau(T(f \circ g)X_1, \dots, T(f \circ g)X_r) \\ &= \tau(Tf(Tg(X_1)), \dots, Tf(Tg(X_r))) \\ &= (f^*\tau)(Tg(X_1), \dots, Tg(X_r)) \\ &= (g^*(f^*\tau))(X_1, \dots, X_r). \end{aligned}$$
>
> So $(f \circ g)^*\tau = g^*(f^*\tau)$.

3. Given $f \in C^\infty(M)$, define $\tau_f(X, Y) = XYf$ for $X, Y \in \mathfrak{X}(M)$. Show that $\tau_f$ is bilinear in $X$ and $Y$. Is $\tau_f$ a tensor field?

> **Solution:** Suppose $X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(M)$ and $c \in \mathbb{R}$. Then
>
> $$\tau_f(X_1 + cX_2, Y) = (X_1 + cX_2)Yf = X_1Yf + cX_2Yf = \tau_f(X_1, Y) + c\tau_f(X_2, Y)$$
>
> and
>
> $$\tau_f(X, Y_1 + cY_2) = X(Y_1 + cY_2)f = XY_1f + cXY_2f = \tau_f(X, Y_1) + c\tau_f(X, Y_2).$$
>
> So $\tau_f$ is bilinear. But if $\phi \in C^\infty(M)$, we have:
>
> $$\tau_f(X, \phi Y) = X(\phi Y)f = (X\phi)(Yf) + \phi XYf = \phi\tau_f(X, Y) + (X\phi)(Yf).$$
>
> For $\tau_f$ to define a tensor, we must have $(X\phi)(Yf) = 0$ for all $X, Y$ and $\phi$. One can choose $X$ and $\phi$ so that $X\phi$ is nonzero at any particular point of $M$ (for example, $X = \frac{\partial}{\partial x}$ and $\phi = x$ in some local coordinate system). Then we must have $Yf \equiv 0$ for all $Y$. This only happens if $f$ is a constant function, and in that case $\tau_f \equiv 0$ is the zero tensor. If $f$ is not constant, $\tau_f$ does not define a tensor.

4. If you rescale a Riemannian metric, what effect does this have on distances between points?

   A Riemannian manifold $(M, g)$ has a metric $d$ giving the distance between any two points in $M$. For $\lambda > 0$, $\lambda g$ is a Riemannian metric on $M$, and say $(M, \lambda g)$ has metric $d_\lambda$. Express $d_\lambda(x, y)$ in terms of $d(x, y)$ and $\lambda$.

**Solution:** For any curve $\gamma : [a, b] \to M$ which joins $x$ and $y$, we have

$$\text{len}_{\lambda g}(\gamma) = \int_a^b \sqrt{\lambda g(\gamma'(t), \gamma'(t))} dt = \sqrt{\lambda} \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} dt = \sqrt{\lambda}\, \text{len}_g(\gamma)$$

So the distance $d_\lambda(x, y) = \inf_\gamma \text{len}_{\lambda g}(\gamma) = \sqrt{\lambda} \inf_\gamma \text{len}_g(\gamma) = \sqrt{\lambda} d(x, y)$.

5. Let $M$ be an $m$-manifold, and $x \in M$. Show there is a Riemannian metric $g$ on $M$ and a neighborhood $U$ of $x$ so that $(U, g|_U)$ is isometric to a subset of $\mathbb{R}^n$ with the Euclidean metric.

**Solution:** Let $(V, \varphi)$ be a chart containing $x$, and let $U$ be a neighborhood of $x$ with $\bar{U} \subset V$. Choose a cutoff function $f : V \to \mathbb{R}$ with $0 \le f \le 1$, $f$ supported on $V$, and $f \equiv 1$ on $U$. Let $h$ be any metric on $M$ and let $e$ be the pullback by $\varphi$ of the Euclidean metric on $\varphi(V)$. Put $g = fe + (1 - f)h$. Then $g|_U = e$ is isometric to the Euclidean metric as desired.

Since $e$ and $h$ are symmetric two tensors, so is $g$. To see $g$ is a metric, check it is positive definite. Off of $V$, $g \equiv h$ so $g$ is positive definite since $h$ is. For $p \in V$, let $0 \neq X \in T_p(M)$, and put $\epsilon = \min(e(X, X), h(X, X)) > 0$. Then

$$g_p(X, X) = f(p)e(X, X) + (1 - f(p))h(X, X) \ge f(p)\epsilon + (1 - f(p))\epsilon = \epsilon > 0,$$

which shows $g$ is a Riemannian metric.

6. Define the one-form

$$\tau = xdy - ydx + zdw - wdz$$

on the sphere $S^3 = \{(x, y, z, w) | x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4$. Fix $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$. For $t \in [0, 2\pi]$ define a curve $\gamma(t) = (a \cos t, a \sin t, b \cos t, b \sin t) \subset S^3$.

Compute $\displaystyle\int_\gamma \tau$.

Bonus generalization(optional, only slightly harder):

If $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$ then $\gamma(t) = e^{it}(a, b)$ is a circle in $S^3 \subset \mathbb{C}^2$. Compute $\displaystyle\int_\gamma \tau$.

**Solution:** From the curve, $x = a \cos t$ so $dx = -a \sin t dt$, $y = a \sin t$ so $dy = a \cos t dt$, and similarly for $z, w$. Then

$$\int_\gamma \tau = \int_0^{2\pi} a^2 \cos^2 t + a^2 \sin^2 t + b^2 \cos^2 t + b^2 \sin^2 t dt = \int_0^{2\pi} a^2 + b^2 dt = \int_0^{2\pi} 1 dt = 2\pi.$$

The bonus generalization also gives $2\pi$.