1. Suppose g_1, g_2 are Riemannian metrics on M. Show that $g = \lambda g_1 + (1-\lambda)g_2$ is a Riemannian metric for any $\lambda \in [0,1]$.

Solution: By linearity, g is a symmetric 2-tensor. We need to show g is positive definite. If X is a nonzero tangent vector, $g_1(X,X) > 0$ and $g_2(X,X) > 0$ so that

$$g(X,X) = \lambda g_1(X,X) + (1-\lambda)g_2(X,X) > 0$$

since $\lambda, 1-\lambda$ are non-negative and at least one of them is non-zero.

2. Suppose M is orientable. Given $a \in \mathbb{R}$, show there is a compactly supported n-form α on M with $\int_M \alpha = a$.

Solution: Let ω be an orientation n-form on M, let (U, φ) be any positively oriented chart, and let $\rho \in C^\infty(M)$ be a bump function with compact support in U. Since (U,φ) is positively oriented, $\varphi^*\omega = f dx^1 \wedge \cdots \wedge dx^n$ for some function $f > 0$. The n-form $\rho\omega$ is compactly supported on U. Put $C = \int_M \rho\omega$. Then

$$C = \int_M \rho\omega = \int_{\varphi(U)} \rho(\phi^{-1}(\mathbf{x}))f(\mathbf{x})d\mathbf{x} > 0.$$

Put $\alpha = \frac{a}{C}\rho\omega$, so $\int_M \alpha = a$.

3. Let $\mathbb{H}$ be Hyperbolic space: the upper half plane $\{(x,y)|y>0\}$ with metric $ds^2 = \frac{1}{y^2}\left(dx^2+dy^2\right)$. Show that each of these three diffeomorphisms is an isometry:

- $(x,y) \to (ax, ay)$, $a > 0$.
- $(x,y) \to (x+b, y)$, $b \in \mathbb{R}$.
- $(x,y) \to \frac{1}{x^2+y^2}(-x, y)$.

(These correspond to the Möbius transformations $z \to az$, $z \to z+b$, $z \to -1/z$, and together generate the group $SL_2(\mathbb{R})$).

Solution:

- $\frac{1}{y^2}\left(dx^2+dy^2\right) \to \frac{1}{(ay)^2}\left((d(ax))^2 + (d(ay))^2\right) = \frac{1}{a^2y^2}a^2\left(dx^2+dy^2\right) = \frac{1}{y^2}\left(dx^2+dy^2\right)$
- $\frac{1}{y^2}\left(dx^2+dy^2\right) \to \frac{1}{y^2}\left((d(x+b))^2 + dy^2\right) = \frac{1}{y^2}\left(dx^2+dy^2\right)$
- Compute

$$d\left(\frac{-x}{x^2+y^2}\right) = \frac{(x^2-y^2)dx + 2xydy}{(x^2+y^2)^2}, \quad d\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2-y^2)dy - 2xydx}{(x^2+y^2)^2}$$

so that

$$\frac{1}{y^2}\left(dx^2+dy^2\right) \to \frac{(x^2+y^2)^2}{y^2}\frac{\left((x^2-y^2)dx+2xydy\right)^2+\left((x^2-y^2)dy-2xydx\right)^2}{(x^2+y^2)^4}$$
$$=\frac{1}{y^2}\frac{\left((x^2-y^2)^2+4x^2y^2\right)(dx^2+dy^2)}{(x^2+y^2)^2}=\frac{1}{y^2}\left(dx^2+dy^2\right).$$

4. Let $\omega = ydx - xdy \in \Omega^1(\mathbb{R}^2)$. Let $\sigma = \frac{1}{x^2+y^2}\omega \in \Omega^1(\mathbb{R}^2 - \{0\})$.
   (a) Is ω closed? Is ω exact?
   (b) Is σ closed? Is σ exact?

   Prove your answers.

Solution:

(a) ω is not closed (and therefore cannot be exact).

Check: $d\omega = dy \wedge dx - dx \wedge dy = -2dx \wedge dy \neq 0$.

(b) σ is closed:

$$\begin{aligned} d\sigma &= d\frac{y}{x^2+y^2}\wedge dx + d\frac{-x}{x^2+y^2}\wedge dy \\ &= \frac{(x^2-y^2)dy-2xydx}{(x^2+y^2)^2}\wedge dx + \frac{(x^2-y^2)dx+2xydy}{(x^2+y^2)^2}\wedge dy \\ &= \frac{1}{(x^2+y^2)^2}\left((x^2-y^2)dy\wedge dx + (x^2-y^2)dx\wedge dy\right) \\ &= 0 \end{aligned}$$

However, σ is not exact. Consider the curve $c(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$. Then

$$\int_c \sigma = \int_0^{2\pi} (\sin t)(-\sin t dt) - (\cos t)(\cos t dt) = \int_0^{2\pi} -1dt = -2\pi.$$

An exact form would have integral 0 around any closed curve.

5. Let α, β be k-forms on a smooth n-manifold M. Let S be a k-dimensional submanifold of M (without boundary). If $[\alpha] = [\beta] \in H^k(M)$, (i.e. α and β represent the same cohomology class) then show that

$$\int_S \alpha = \int_S \beta$$

Solution: Write $\alpha = \beta + d\tau$ for some $k-1$ form τ. Then applying Stokes' Theorem,

$$\int_S \alpha = \int_S \beta + d\tau = \int_S \beta + \int_S d\tau = \int_S \beta + \int_{\partial S} \tau = \int_S \beta$$

6. (a) For a smooth compact oriented M with volume form μ, prove

$$\int_M (\operatorname{Div} X)\mu = \int_{\partial M} \iota_X \mu$$

for any vector field X on M.

(b) Suppose, additionally, that M has no boundary. Prove that for any vector field X, there must be a point on M where $\operatorname{Div} X$ vanishes.

(c) Give an example of a manifold M and a vector field X where $\operatorname{Div} X$ is nowhere zero.

Solution:

(a)

$$\begin{aligned}
\int_M (\operatorname{Div} X)\mu &= \int_M \mathcal{L}_X \mu \\
&= \int_M d\iota_X \mu + \iota_X d\mu && \text{(Cartan's Formula)} \\
&= \int_M d\iota_X \mu && (d\mu = 0) \\
&= \int_{\partial M} \iota_X \mu && \text{(Stokes' Theorem)}
\end{aligned}$$

(b) From part (a) and $\partial M = \emptyset$, $\int_M (\operatorname{Div} X)\mu = 0$. Assume M is connected. Suppose $\operatorname{Div} X$ never vanishes. Since $\operatorname{Div} X$ is continuous and M is connected, $\operatorname{Div} X$ is strictly positive or strictly negative on M. Then $\int_M (\operatorname{Div} X)\mu$ cannot be zero, a contradiction. If M is not connected, the above argument shows $\operatorname{Div} X$ vanishes on any connected component of M.

(c) One example is the vector field $X = x\frac{\partial}{\partial x}$ on $M = \mathbb{R}$ (the radial field on $\mathbb{R}^n$ for any n also works). Here, $\mu = dx$ and

$$\mathcal{L}_X \mu = d(\iota_X \mu) = d(x) = \mu$$

so that $\operatorname{Div} X \equiv 1$.