Final Exam

1. Suppose g_1, g_2 are Riemannian metrics on M. Show that $g = \lambda g_1 + (1-\lambda) g_2$ is a Riemannian metric for any $\lambda \in [0,1]$.

2. Suppose M is orientable. Given $a \in \mathbb{R}$, show there is a compactly supported n-form α on M with $\int_M \alpha = a$.

3. Let $\mathbb{H}$ be Hyperbolic space: the upper half plane $\{(x,y) | y > 0\}$ with metric $ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right)$. Show that each of these three diffeomorphisms is an isometry:

   - $(x,y) \to (ax, ay)$, $a > 0$.
   - $(x,y) \to (x+b, y)$, $b \in \mathbb{R}$.
   - $(x,y) \to \frac{1}{x^2+y^2}(-x, y)$.

   (These correspond to the Möbius transformations $z \to az$, $z \to z+b$, $z \to -1/z$, and together generate the group $SL_2(\mathbb{R})$).

4. Let $\omega = y dx - x dy \in \Omega^1(\mathbb{R}^2)$. Let $\sigma = \frac{1}{x^2+y^2}\omega \in \Omega^1(\mathbb{R}^2 - \{0\})$.

   (a) Is ω closed? Is ω exact?

   (b) Is σ closed? Is σ exact?

   Prove your answers.

5. Let α, β be k-forms on a smooth n-manifold M. Let S be a k-dimensional submanifold of M (without boundary). If $[\alpha] = [\beta] \in H^k(M)$, (i.e. α and β represent the same cohomology class) then show that
$$\int_S \alpha = \int_S \beta$$

6. (a) For a smooth compact oriented M with volume form μ, prove
$$\int_M (\operatorname{Div} X)\mu = \int_{\partial M} \iota_X \mu$$
   for any vector field X on M.

   (b) Suppose, additionally, that M has no boundary. Prove that for any vector field X, there must be a point on M where $\operatorname{Div} X$ vanishes.

   (c) Give an example of a manifold M and a vector field X where $\operatorname{Div} X$ is nowhere zero.