On this exam, M will always be a smooth m-dimensional manifold.

1. Give an example of a topological space X which is locally Euclidean but not a manifold. (Define the topology on X and show both claims)

> **Solution:** Let X be the union of $\mathbb{R}$ with a single point set $\{0^*\}$. The open sets of X are the open sets of $\mathbb{R}$ together with sets of the form $U^* = (U - 0) \cup 0^*$, where U is any open neighborhood of 0 in $\mathbb{R}$.
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> X is not Hausdorff, so it is not a manifold (any neighborhood U of 0 intersects the corresponding neighborhood U^* of 0^*).
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> X is locally Euclidean. The identity map on $\mathbb{R}$ gives a local homeomorphism to $\mathbb{R}$ for all $x \in \mathbb{R}$, and the map which is the identity on $\mathbb{R}-0$ and sends 0^* to $0 \in R$ is a homeomorphism of a neighborhood of 0^* with $\mathbb{R}$.

2. With $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2+y^2+z^2 = 1\}$, let $C \subset S^2$ consist of the three circles $z = \frac{1}{2}$, $z = 0$, $z = -\frac{1}{2}$. Accurately sketch the image of C under stereographic projection from the north pole $(0, 0, 1)$.

> **Solution:** Stereographic projection takes (x, y, z) to $\frac{1}{1-z}(x, y) = (X, Y)$. The circle corresponding to height z has $x^2 + y^2 = 1 - z^2$, so its image satisfies
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> $$X^2 + Y^2 = \frac{1}{(1-z)^2}(x^2 + y^2) = \frac{1-z^2}{(1-z)^2} = \frac{1+z}{1-z}.$$
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> Thus the image of C consists of three circles in the plane, centered at the origin, with radii $\sqrt{3}$, 1 and $\frac{1}{\sqrt{3}}$ respectively.

3. Given smooth functions $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ and distinct points $p, q \in M$, show that there is a smooth function $h : M \to \mathbb{R}$ so that $h \equiv f$ in a neighborhood of p, and $h \equiv g$ in a neighborhood of q.

> **Solution:** Choose open neighborhoods U_p and U_q of p and q with $U_p \cap U_q = 0$ (M is Hausdorff). Now choose smaller V_p, V_q open neighborhoods of p and q, with compact closures, and with $\bar{V_p} \subset U_p$ and $\bar{V_q} \subset U_q$. Specifically, one could choose a chart around p and let V_p be a ball centered at p with sufficiently small radius.
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> Now let φ_p be a smooth cutoff function which is 1 on V_p and has compact support contained in U_p. Extend φ_p by 0 to a smooth function on M, and note that $\varphi_p \equiv 0$ on U_q. Similarly define φ_q which is 1 on V_q and 0 on U_p.
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> Put $h = \varphi_p f + \varphi_q g$, a smooth function with $h \equiv f$ on V_p and $h \equiv g$ on V_q.

4. Let $\Delta \subset M \times M$ be the diagonal, $\Delta = \{(p, p) | p \in M\}$. Describe charts on Δ that make Δ an m-manifold diffeomorphic to M. (You don't need to prove anything, just define the charts).

Solution: For each chart (U, φ) on M, let $U_\Delta = \{(x, x) \in \Delta | x \in U\}$ and define $\varphi_\Delta : U_\Delta \to \mathbb{R}^m$ by $\varphi_\Delta(x, x) = \varphi(x)$. Then $(U_\Delta, \varphi_\Delta)$ are charts on Δ.

5. Let $f : M \to \mathbb{R}$ be a smooth function, and suppose f takes its maximum value at $p \in M$. For any $X_p \in T_pM$, show that $X_p f = 0$.

Solution: Let $c : (-\epsilon, \epsilon) \to M$ be a curve representing X_p with $c(0) = p$. Then $f \circ c : (-\epsilon, \epsilon) \to \mathbb{R}$ is a smooth function which takes its maximum value at 0. From calculus, $(f \circ c)'(0) = 0$, and by definition, $X_p f = (f \circ c)'(0)$.

6. Show that

$$f([x : y]) = \frac{xy}{x^2 + y^2}$$

is well defined as a function $f : \mathbb{R}P^1 \to \mathbb{R}$, where $[x : y]$ are homogenous coordinates on projective space.

Find the maximum of f on $\mathbb{R}P^1$ (you may use problem 5).

Solution: For c a scalar,

$$f([cx : cy]) = \frac{c^2 xy}{c^2 x^2 + c^2 y^2} = \frac{xy}{x^2 + y^2} = f([x : y])$$

so f is well defined as a function on $\mathbb{R}P^1$.

f does have a maximum on $\mathbb{R}P^1$ since $\mathbb{R}P^1$ is compact.

First, note that $f([1 : 0]) = 0$, and we will see this is not the maximum of f.

Now, choose coordinates on $\mathbb{R}P^1 - [1 : 0]$ by $[x : 1] \to x$. In these coordinates (this coordinate?),

$$f(x) = f[x : 1] = \frac{x}{x^2 + 1}$$

By problem 5, if f has a maximum at x then $\frac{\partial}{\partial x} f\Big|_x = 0$. Compute

$$\frac{\partial}{\partial x} f\Big|_x = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

which vanishes when $1 - x^2 = 0$ or when $x = \pm 1$.

Now check $f([1 : 1]) = \frac{1}{2}$ and $f([-1 : 1]) = -\frac{1}{2}$, so we find the maximum value of f is $\frac{1}{2}$ at the point $[1 : 1]$.