Math 641

Midterm Exam

10/24/12

On this exam, M will always be a smooth m-dimensional manifold.

1. Give an example of a topological space X which is locally Euclidean but not a manifold. (Define the topology on X and show both claims)

2. With $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$, let $C \subset S^2$ consist of the three circles $z = \frac{1}{2}$, $z = 0$, $z = -\frac{1}{2}$. Accurately sketch the image of C under stereographic projection from the north pole $(0, 0, 1)$.

3. Given smooth functions $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ and distinct points $p, q \in M$, show that there is a smooth function $h : M \to \mathbb{R}$ so that $h \equiv f$ in a neighborhood of p, and $h \equiv g$ in a neighborhood of q.

4. Let $\Delta \subset M \times M$ be the diagonal, $\Delta = \{(p, p) | p \in M\}$. Describe charts on Δ that make Δ an m-manifold diffeomorphic to M. (You don't need to prove anything, just define the charts).

5. Let $f : M \to \mathbb{R}$ be a smooth function, and suppose f takes its maximum value at $p \in M$. For any $X_p \in T_pM$, show that $X_p f = 0$.

6. Show that

$$f([x : y]) = \frac{xy}{x^2 + y^2}$$

is well defined as a function $f : \mathbb{R}P^1 \to \mathbb{R}$, where $[x : y]$ are homogenous coordinates on projective space.

Find the maximum of f on $\mathbb{R}P^1$ (you may use problem 5).