1. For M a smooth m-manifold, let $v \in T_p(M)$. Show that there is a smooth vector field X on M with $X(p) = v$.

Solution: Choose coordinates $x_1, \ldots, x_m$ on an open neighborhood U of p. Write $v = \sum_i v_i \frac{\partial}{\partial x_i}$ (here v_i are simply numbers). Define a vector field Y on U by $Y = \sum_i v_i \frac{\partial}{\partial x_i}$, so clearly $Y_p = v$. Choose open neighborhoods $p \in V \subset W \subset U$ so that $\overline{V} \subset U$ and $\overline{W} \subset U$, and let φ be a smooth cutoff function which is 1 on V and 0 outside of W. Define $X = \varphi Y$ on U and $X = 0$ otherwise. X is smooth U and zero outside of W, hence a smooth vector field on all of M, and $X_p = \varphi(p) Y_p = v$.

2. Define three vector fields on $\mathbb{R}^3$:

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}; \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}; \quad Z = \frac{\partial}{\partial z}.$$

Show that $[X, Z] = [Y, Z] = 0$ and that $[X, Y] = Z$.

Solution: This is a straightforward computation. These aren't just random fields. The Heisenberg group $\mathcal{H}$ consists of 3x3 matrices of the form $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, and identifying $\mathcal{H}$ with $\mathbb{R}^3$, the fields X, Y, and Z are a basis for the vector fields which are invariant under the group multiplication.

3. Suppose M is a hypersurface ($n-1$-dimensional embedded manifold) in $\mathbb{R}^n$. Let N be the be *normal bundle* over M: The vector bundle whose fiber at $p \in M$ is the one-dimensional space of normal vectors to M at p. Show that N is trivial if and only if M is orientable.

Solution: If M is orientable then there is a unit normal vector field $X : M \to N$. This is a nonzero section of N, so N is trivial. Conversely, if N is trivial, there is a nonzero section $X : M \to N$. Then $\frac{X}{||X||}$ is a unit normal vector field and so M is orientable.

4. Let $c : \mathbb{R} \to \mathbb{R}^2$ be any smooth curve in the plane. Show that for all $\varepsilon > 0$, there is $x \in \mathbb{R}^2$ with $||x|| < \varepsilon$ so that x is not in the image of the curve c.

Solution: Sard's Theorem says that the set χ of critical values of c has measure 0 in $\mathbb{R}^2$. Then χ does not contain the ball $B(0, \varepsilon)$, so choose $x \in B(0, \varepsilon) - \chi$. The point x is a regular value of c, but since $\dim T\mathbb{R} = 1 < 2 = \dim T\mathbb{R}^2$, Tc is never surjective, so x is not in the image of c.

5. Let $M(2)$ denote the space of 2×2 matrices with real entries. Let

$$N = \{A \in M(2) | A \neq 0, \det(A) = 0\}.$$

Show that N is a manifold.

Solution:

Way 0: The map det is given by $\det A = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$. This has tangent map $T\det = (d, -c, -b, a) : T\mathbb{R}^4 \to T\mathbb{R}$. This has rank 1 unless A is the zero matrix. Now $M^* = M(2) - \{0\}$ is a manifold since it is an open subset of $M(2)$, and 0 is a regular value of $\det\big|_{M^*}$, so $\det^{-1}(0) = N$ is a manifold.

Way 1: Let U_ℓ be the set of matrices in N with nonzero left column, and U_r be the set of matrices in N with nonzero right column. Note that $N = U_\ell \cup U_r$. For $A \in U_\ell$, write $A = \begin{pmatrix} x & \lambda x \\ y & \lambda y \end{pmatrix}$ (which we can do because the columns of A are linearly dependent). Put $\phi_\ell(A) = (x, y, \lambda)$. Similarly, for $A \in U_r$, write $A = \begin{pmatrix} \lambda x & x \\ \lambda y & y \end{pmatrix}$ and put $\phi_r(A) = (x, y, \lambda)$. On $U_\ell \cap U_r$, the change of coordinates map is given by $(\phi_r^{-1} \circ \phi_\ell)(x, y, \lambda) = (\lambda x, \lambda y, \lambda^{-1})$, which is smooth. The inverse $\phi_\ell^{-1} \circ \phi_r$ has the same formula and is also smooth. Then (U_ℓ, φ_ℓ) and (U_r, φ_r) define an atlas on N.

Way 2: For $A \in N$, the kernel of A is a line through the origin. Let U_h be the set of $A \in N$ whose kernel is not horizontal, and U_v be the A with kernel which is not vertical. For $A \in U_h$, let $\theta \in (0, \pi)$ be the angle that $\ker A$ makes with the positive x-axis (well defined on U_h). Let $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation by θ. Then $AR_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$, so $AR_\theta = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$ and define $\varphi_h(A) = (x, y, \theta)$. Note $\begin{pmatrix} x \\ y \end{pmatrix} = AR_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Similarly define $\varphi_v(A)$ on U_v, except $\theta \in (-\pi/2, \pi/2)$. When $\ker A$ has positive slope, $\varphi_h(A) = \varphi_v(A)$ so the coordinate change is just the identity. When $\ker A$ has negative slope, if $\varphi_h(A) = (x, y, \theta)$ then $\varphi_v(A) = (-x, -y, \theta - \pi)$ since $R_{\theta-\pi} = -R_\theta$. Then (U_h, φ_h) and (U_v, φ_v) define an atlas on N.

Note: Way 1 and way 2 are reminescent of putting stereographic and angular coordinates on a circle, respectively. In both cases, it's easy to see that the set of matrices in N with a fixed λ or θ form a two dimensional vector space, so that N is a vector bundle over the circle. N is a trivial bundle over S^1 (show it!) so that N is diffeomorphic to $\mathbb{R}^2 \times S^1$.

Bonus: Generalize these results to $N \subset M(n)$, the set of $n \times n$ matrices with one dimensional kernel. What dimension is N? Generally, N is a bundle over $\mathbb{R}P^{n-1}$ with projection $\pi : N \to \mathbb{R}P^{n-1}$ given by $\pi(A) = \ker A$. Is this a trivial bundle?

6. Show that there is no submersion $S^1 \to \mathbb{R}$. Is there a submersion $S^1 \times \mathbb{R} \to \mathbb{R}^2$?

Solution: Suppose $f : S^1 \to \mathbb{R}$ is smooth. Let $y = \sup_{t \in S^1} f(t)$. Since S^1 is compact, there is $t \in S^1$ with $f(t) = y$. In local coordinates near t, f has a maximum at t so $T_t f = 0$ and $T_t f$ is not surjective onto $T_y\mathbb{R}$. So f is not a submersion.

One submersion from $S^1 \times \mathbb{R} \to \mathbb{R}^2$ is given by $f(\theta, r) = (r\cos(\theta), r\sin(\theta))$.

7. The Mobius strip M is the quotient of $\mathbb{R} \times [-\pi/2, \pi/2]$ by the equivalence $(\theta, \varphi) \sim (\theta + 2\pi, -\varphi)$. Define $f : M \to \mathbb{C}^2$ by $f(\theta, \varphi) = (\cos(\varphi)e^{i\theta}, \sin(\varphi)e^{i\theta/2})$.
   (a) Show that f is well defined as a function on M.
   (b) Show that f is an embedding.
   (c) Show that the image of f lies in the unit sphere $S^3 \subset \mathbb{C}^2$.

   Remark: This embedding is cool because the boundary of M is sent to a perfect circle in S^3. Applying stereographic projection, one gets an embedding of M into $\mathbb{R}^3$ with circular boundary.

Solution:

(a)

$$\begin{aligned} f(\theta + 2\pi, -\varphi) &= (\cos(-\varphi)e^{i(\theta+2\pi)}, \sin(-\varphi)e^{i(\theta+2\pi)/2}) \\ &= (\cos(\varphi)e^{i\theta}, -\sin(\varphi)e^{i\theta/2}e^{i\pi}) = f(\theta, \varphi) \end{aligned}$$

So f is well defined.

(b) First, see f is injective. Suppose $f(\theta, \varphi) = (z, w)$ with $\theta \in [0, 2\pi)$. We know $\sin\varphi = \pm|w|$. Since $\operatorname{Im} e^{i\theta/2} \geq 0$, the sign on $\sin\varphi$ must match the sign of $\operatorname{Im} w$, which means $\varphi = \arcsin(\pm|w|)$. Once φ is known, θ can be computed from $z/\cos(\varphi)$ or $w/\sin(\varphi)$, whichever doesn't involve division by zero.

Next, show f is an immersion:

$$Tf = \begin{pmatrix} i\cos(\varphi)e^{i\theta} & -\sin(\varphi)e^{i\theta} \\ \frac{i}{2}\sin(\varphi)e^{i\theta/2} & \cos(\varphi)e^{i\theta/2} \end{pmatrix}$$

so $\det Tf = ie^{3i\theta/2}(\cos^2(\varphi) + \frac{1}{2}\sin^2(\varphi)) \neq 0$. So, Tf has rank 2 everywhere and f is an immersion.

(c) For any (θ, ϕ), $|\cos(\varphi)e^{i\theta}|^2 + |\sin(\varphi)e^{i\theta/2}|^2 = \cos^2(\varphi) + \sin^2(\varphi) = 1$, so the image of f is on the unit sphere in C^2.

8. Let $M(n)$ denote the vector space of $n \times n$ matrices. Since $M(n)$ is a vector space, the tangent space to $M(n)$ at the identity is naturally identified with $M(n)$. Let $O(n) \subset M(n)$ be the orthogonal matrices. Show any tangent vector to $O(n)$ at the identity is a skew-symmetric matrix.

Solution: Let $S \in T_I O(n)$. Let $A(t)$ be a curve in $O(n)$ with $A(0) = I$ with $A'(0) = S$. Then $I = AA^T$ for all t. Taking the derivative of both sides at $t = 0$,

$$0 = \frac{d}{dt} AA^T\Big|_{t=0} = A'A^T + A(A^T)'\Big|_{t=0} = A'(0)I + I(A')^T(0) = A'(0) + (A'(0))^T = S + S^T.$$

Since $S + S^T = 0$, S is skew-symmetric.