1. For M a smooth m-manifold, let $v \in T_p(M)$. Show that there is a smooth vector field X on M with $X(p) = v$.

2. Define three vector fields on $\mathbb{R}^3$:

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}; \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}; \quad Z = \frac{\partial}{\partial z}.$$

   Show that $[X, Z] = [Y, Z] = 0$ and that $[X, Y] = Z$.

3. Suppose M is a hypersurface ($n-1$-dimensional embedded manifold) in $\mathbb{R}^n$. Let N be the be *normal bundle* over M: The vector bundle whose fiber at $p \in M$ is the one-dimensional space of normal vectors to M at p. Show that N is trivial if and only if M is orientable.

4. Let $c : \mathbb{R} \to \mathbb{R}^2$ be any smooth curve in the plane. Show that for all $\varepsilon > 0$, there is $x \in \mathbb{R}^2$ with $||x|| < \varepsilon$ so that x is not in the image of the curve c.

5. Let $M(2)$ denote the space of 2×2 matrices with real entries. Let

$$N = \{A \in M(2) | A \neq 0, \det(A) = 0\}.$$

   Show that N is a manifold.

6. Show that there is no submersion $S^1 \to \mathbb{R}$. Is there a submersion $S^1 \times \mathbb{R} \to \mathbb{R}^2$?

7. The Mobius strip M is the quotient of $\mathbb{R} \times [-\pi/2, \pi/2]$ by the equivalence $(\theta, \varphi) \sim (\theta + 2\pi, -\varphi)$. Define $f : M \to \mathbb{C}^2$ by $f(\theta, \varphi) = (\cos(\varphi)e^{i\theta}, \sin(\varphi)e^{i\theta/2})$.
   (a) Show that f is well defined as a function on M.
   (b) Show that f is an embedding.
   (c) Show that the image of f lies in the unit sphere $S^3 \subset \mathbb{C}^2$.

   Remark: This embedding is cool because the boundary of M is sent to a perfect circle in S^3. Applying stereographic projection, one gets an embedding of M into $\mathbb{R}^3$ with circular boundary.

8. Let $M(n)$ denote the vector space of $n \times n$ matrices. Since $M(n)$ is a vector space, the tangent space to $M(n)$ at the identity is naturally identified with $M(n)$. Let $O(n) \subset M(n)$ be the orthogonal matrices. Show any tangent vector to $O(n)$ at the identity is a skew-symmetric matrix.