The deRham Cohomology of S^n.

Given that 1) cohomology is a homotopy invariant, and 2) the Poincare lemma, compute the deRham cohomology of S^n.

Proceed by induction on n, the $n = 1$ case computed directly by integration.

Let $S^n = U \cup V$ with $U \simeq \mathbb{R}^n, V \simeq \mathbb{R}^n$, and $U \cap V \simeq S^{n-1}$. For example, U and V could be S^n with the North and South pole removed, respectively.

Let $\omega \in \bigwedge^k(S^n)$ be a closed k-form.

- Case $1 < k < n$:

  Since $H^k(\mathbb{R}^n) = 0$, there is $\alpha \in \bigwedge^{k-1}(U)$ and $\beta \in \bigwedge^{k-1}(V)$ with $d\alpha = \omega|_U$ and $d\beta = \omega|_V$.

  On $U \cap V$, $d(\alpha - \beta) = 0$. Since $H^{k-1}(S^{n-1}) = 0$, there is $\tau \in \bigwedge^{k-2}(U \cap V)$ with $d\tau = (\alpha - \beta)|_{U \cap V}$.

  Now let f_U, f_V be a partition of unity subordinate to U, V. Then on $U \cap V$, $d(f_U\tau) + d(f_V\tau) = \alpha - \beta$.

  Since $f_U\tau$ extends (by 0) to a form on V, $\beta + d(f_U\tau)$ is defined on V. Since $f_V\tau$ extends to a form on U, $\alpha - d(f_V\tau)$ is defined on U.

  Put $\sigma = \beta + d(f_U\tau) = \alpha - d(f_V\tau)$. σ is defined on all of S^n, and $d\sigma = \omega$, so that ω is exact.

- Case $k = 1$:

  When $k = 1$, α and β as above are functions with $d(\alpha - \beta) = 0$, so $\alpha = \beta + C$ for some constant C, so that α extends to a function on all of S^n and $d\alpha = \omega$.

- Case $k = n$: Show that any n form with integral 0 is exact, so that $H^n(S^n) = \mathbb{R}$ generated by the volume form.

  Suppose $\int_{S^n} \omega = 0$. Take α, β as above, and let H, L be the upper and lower hemispheres of S^n. Then using Stokes' theorem,

  $$\int_{S^{n-1}} \alpha - \beta = \int_L d\alpha + \int_H d\beta = \int_{S^n} \omega = 0$$

  Then $\alpha - \beta$ is exact, and the proof proceeds as in the $1 < k < n$ case.