Complex Variables - Review Questions

1. The integer 53 is the sum of two squares since $53 = 2^2 + 7^2$. Can every integer be expressed as the sum of two squares?

   (a) Prove this fact: If two integers can be expressed as the sum of two squares, then so can their product. More precisely, if $M = a^2 + b^2$ and $N = c^2 + d^2$ then there are p and q with $MN = p^2 + q^2$ (where all these symbols denote integers).

   [Hint: Consider $z = a + bi$, $w = c + di$.]

   (b) Write 1378 (=26×53) as the sum of two squares.

2. (a) Suppose a bug is walking in the plane. The bug walks one unit east, then turns 90° left, and walks 1/2 unit forwards, then turns 90° left again, and walks 1/4 unit forwards, and so on. At each step, the bug turns 90° left and walks half as far as it did on the previous step. Where will the bug end up?

   (b) Generalize so that at each step, the bug turns θ to the left and walks $r < 1$ times as far as it did on the previous step.

3. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic and preserves parallel lines. That is, the image of any pair of parallel lines is still a pair of parallel lines. Prove that f is linear, i.e. $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

   [Hint: Suppose first that f takes any pair of horizontal lines to a pair of horizontal lines, and see what effect this has on the Cauchy-Riemann equations.]

4. Recall the binomial coefficients $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ can be used for the binomial expansion:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

   (a) For Γ any positively oriented simple closed curve around the origin, prove that

$$\binom{n}{r} = \frac{1}{2\pi i} \int_\Gamma \frac{(1+w)^n}{w^{r+1}} dw.$$

   [Hint: Let $f(z) = (1+z)^n$ and compute $f^{(r)}(0)$ in two ways.]

   (b) For all $n = 0, 1, 2, \ldots$, prove that

$$\binom{2n}{n} \leq 4^n.$$

   [Hint: Apply part (a) with Γ the unit circle.]