Math 452 - Homework 8 - Due Wednesday, April 7

For a non-zero function f, the *logarithmic derivative* of f is defined to be $\ell f = \dfrac{f'}{f}$.

1. Suppose $\operatorname{Re} f(z) > 0$ for all z. Prove that $\ell f = \frac{d}{dz} \operatorname{Log} f(z)$.

2. (a) (Product rule) If $f_1, \ldots, f_n$ are nonzero functions, prove that

$$\ell\left(\prod_{i=1}^{n} f_i\right) = \sum_{i=1}^{n} \ell f_i$$

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   (b) (Chain rule) If f and g are nonzero functions, prove that

$$\ell(f \circ g) = (\ell f) \circ g \cdot g'$$

3. Suppose f is entire and nonzero, and let $n \in \mathbb{N}$. Prove there is an entire function h with $h^n = f$, and prove that $\ell f = n\ell h$.

4. Use the factorization of $\sin \pi z$ to compute

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right).$$

5. Find an infinite product factorization for $\sinh(z)$.

6. Compute the logarithmic derivative of $f(z) = \sin \pi z$ and its factorization. Then differentiate term-by-term to prove the following:

$$\pi^2 \csc^2 \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

   Explain why this proves that $\sin^2(\pi z)$ has period 1.

7. Prove

$$\lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n = e^{-t}.$$

8. The Mellin Transform of a function f is defined as

$$(\mathcal{M} f)(z) = \int_0^{\infty} x^z f(x) \frac{dx}{x}.$$

   Note that $\Gamma(z)$ is the Mellin transform of e^{-x}.

   Suppose there are real constants $a < b$ so that $|f(x)| < Cx^{-a}$ for $0 < x < \epsilon$, and $|f(x)| < Cx^{-b}$ for $x > N$, for some positive C. Prove that the Mellin transform of f converges for $a < \operatorname{Re} z < b$. This is called the *strip of convergence*.

9. Suppose the Mellin transform of f exists, and suppose there is $a \in \mathbb{C}$ with $g(x) = f(ax)$. Prove that

$$(\mathcal{M}g)(z) = a^{-z}(\mathcal{M}f)(z).$$

10. Calculate $\Gamma(n)$ for $n \in \mathbb{N}$, using the definition of $\Gamma(z)$ as an improper integral and integrating by parts.

11. Prove that

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges absolutely for all z.

12. Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n)$. Prove that $\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}$, and use that to show that a_n is a positive, decreasing sequence. This means that $\gamma = \lim_{n\to\infty} a_n$ exists. The value γ is known as Euler's constant.