Math 452 - Homework 7 - Due Wednesday, March 24

Do Conway, Pg 173 # 1, 4, 5, 6, 7, 8.

1. For each sequence below, determine the convergence or divergence of $\prod_{n=1}^{\infty} z_n$:

   (a) $z_n = 1 - \dfrac{1}{n+1}$

   (b) $z_n = 1 - \dfrac{1}{n^2+1}$

   (c) $z_n = \cos(\pi n)$

   (d) $z_n = \sin(\pi/n)$ (for $n \geq 2$)

   (e) $z_n = \cos(\pi/n)$ (for $n \geq 3$)

2. (a) Show that $(1-\frac{1}{4})(1-\frac{1}{9})(1-\frac{1}{16})(1-\frac{1}{25})\cdots$ converges absolutely.

   (b) Show that $(1-\frac{1}{2})(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{3})(1-\frac{1}{4})(1+\frac{1}{4})\cdots$ converges, but not absolutely. (Hint: relate to part (a))

3. Show that

$$\prod_{m=1}^{\infty} \left(1 - \frac{1}{m}\right) e^{1/m}$$

   converges absolutely.

4. Let $p_n$ denote the $n^{th}$ prime number ($p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$). Prove that

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^2}\right)^{-1} = \sum_{m=1}^{\infty} \frac{1}{m^2}.$$

   Hint: Expand each term of the left hand side as a geometric series, then multiply term-by-term.

5. For which of the following sequences $a_n(z)$ is the product

$$\prod_{n=1}^{\infty} (1 + a_n(z))$$

   convergent to an entire function $P(z)$?

   (a) $a_n(z) = q^n z$, where $|q| < 1$.

   (b) $a_n(z) = \dfrac{z}{n^2}$.

   (c) $a_n(z) = \dfrac{z}{\log(n+1)}$.