1. $\displaystyle\int_{-1}^{0} \frac{1}{\sqrt[3]{x}} dx$

Solution:

$$\int_{-1}^{0} x^{-1/3} dx = \lim_{b\to 0^-} \frac{3}{2} x^{2/3}\Big|_{-1}^{b} = \frac{3}{2} \lim_{b\to 0^-} b^{2/3} - (-1)^{2/3} = -\frac{3}{2}.$$

2. $\displaystyle\int_{0}^{e} \ln(x) dx$

Solution:

$$\int_{0}^{e} \ln(x) dx = \lim_{a\to 0^+} x\ln(x) - x\Big|_{a}^{e} = e\ln(e) - e = 0$$

Where we need to use the fact that $\lim_{a\to 0^+} a\ln(a) = 0$, which can be shown by writing $a\ln(a) = \frac{\ln(a)}{a^{-1}}$ and applying L'Hopital's rule.

3. $\displaystyle\int_{-1}^{1} \frac{1}{x^2} dx$

Solution: The integral is improper at $x = 0$. The piece from 0 to 1 is given by:

$$\int_{0}^{1} x^{-2} dx = \lim_{a\to 0^+} -x^{-1}\Big|_{a}^{1} = \infty.$$

Regardless of the other piece from -1 to 0 (which is also ∞), this integral diverges.

4. Let $\Gamma(n) = \displaystyle\int_{0}^{\infty} x^{n-1} e^{-x} dx$.
   (a) Find $\Gamma(1)$.
   (b) Show that $\Gamma(n+1) = n\Gamma(n)$.
   (c) Use part (b) to find $\Gamma(6)$.

Solution:

(a) $\Gamma(1) = \displaystyle\int_{0}^{\infty} e^{-x} dx = \lim_{b\to\infty} -e^{-x}\big|_{0}^{b} = \lim_{b\to\infty} -e^{-b} + 1 = 1.$

(b) Integrate by parts with $u = x^n$, $dv = e^{-x}dx$ to get

$$\begin{aligned}\Gamma(n+1) = \int_{0}^{\infty} x^n e^{-x} dx &= \lim_{b\to\infty} \int_{0}^{b} x^n e^{-x} dx \\ &= \lim_{b\to\infty} -x^n e^{-x}\big|_{0}^{b} - \int_{0}^{b} -e^{-x}\cdot n x^{n-1} dx \\ &= \lim_{b\to\infty} -b^n e^{-b} - 0 + n\int_{0}^{\infty} x^{n-1} e^{-x} dx = n\Gamma(n).\end{aligned}$$

(c) $\Gamma(6) = 5\cdot\Gamma(5) = 5\cdot 4\cdot\Gamma(4) = \cdots = 5\cdot 4\cdot 3\cdot 2\cdot\Gamma(1) = 5! = 120.$