Name: ____________________

Math 1520 – Sample Final Exam Solutions

You may use a graphing calculator (TI-83, 84, for example) on this exam, but not one that can perform symbolic integration (TI-89, for example).

There are 15 questions, worth a total of 150 points.

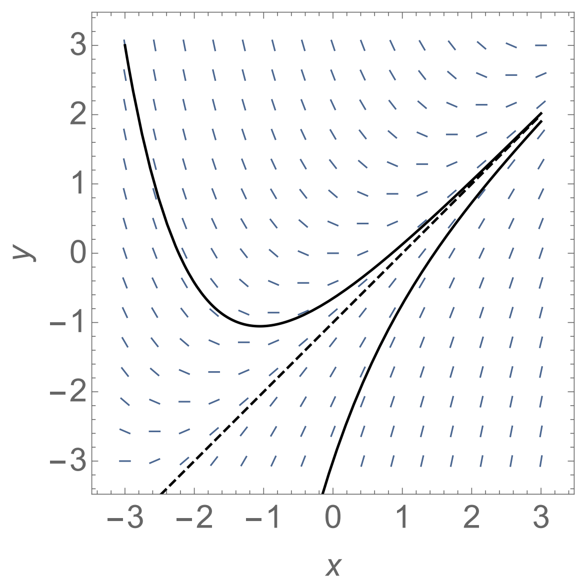
(10) 1. Find the solution to the separable differential equation $\frac{dy}{dx} = x\sqrt{y}$ with initial condition $y = 4$ when $x = 0$.

> **Solution:** Separate: $y^{-1/2}dy = x\ dx$. Integrate: $2y^{1/2} = x^2/2 + C$. When $x = 0$, $2(4)^{1/2} = C$ so $C = 4$ and $y = (\frac{x^2}{4} + 2)^2$.

(10) 2. The slope field for the differential equation $\frac{dy}{dx} = x - y$ is shown below.

(a) Sketch the two solutions which have initial conditions $(-3, 3)$ and $(0, -3)$.

(b) Guess one linear solution to the differential equation and check that it works.



> **Solution:** The solutions are shown in the picture. You can guess the dotted line $y = x - 1$ might be a solution. In that case, $\frac{dy}{dx} = 1 = x - (x - 1) = x - y$.

(10) 3. The Pioneer 10 spacecraft is powered by plutonium radio-thermal generators. The power produced depends directly on the amount of plutonium remaining. The amount of plutonium, P, decays according to the differential equation $\frac{dP}{dt} = -rP$.

(a) Find r, the decay coefficient, given that the half-life of plutonium is 87.72 years.

Solution: $r = \frac{\log(2)}{87.72} \approx 0.0079$

(b) At launch, the power generated was 2580 Watts. How much power was being generated when it sent its last signal in 2003, 31 years after launch?

Solution: Power after 31 years is $2580e^{-r\cdot 31} \approx 2019$ Watts.

(10) 4. The (infinite) region bounded by the curve $y = e^{-x}$, the positive y-axis and the positive x-axis is revolved around the x-axis. Find the volume of this solid of revolution.

Solution:

$$\int_0^\infty \pi \left(e^{-x}\right)^2 dx = -\frac{1}{2}\pi e^{-2x}\Big|_0^\infty = \frac{\pi}{2}$$

(10) 5. Integrate $\displaystyle\int \frac{dx}{x^2 + 2x}$

Solution: Using partial fractions, $\frac{1}{x^2+2x} = \frac{1/2}{x} - \frac{1/2}{x+2}$. Integrate to get $\frac{1}{2}\log\left|\frac{x}{x+2}\right| + C$

(10) 6. Integrate $\displaystyle\int \frac{dx}{x^2 + 2x + 1}$

Solution: With $u = x + 1$:

$$\int \frac{dx}{x^2 + 2x + 1} = \int \frac{dx}{(x+1)^2} = \int u^{-2} du = -u^{-1} + C = \frac{-1}{x+1} + C$$

(10) 7. Integrate $\displaystyle\int \frac{dx}{x^2 + 2x + 2}$

Solution:

$$\int \frac{dx}{x^2 + 2x + 2} = \int \frac{dx}{(x+1)^2 + 1} = \arctan(x+1) + C$$

(10) 8. $2e$ or not $2e$? For each, decide if the value is $2e$ or not $2e$.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n!}$

(b) $\displaystyle\lim_{n\to\infty} 2\left(1+\frac{1}{n}\right)^n$

(c) $\displaystyle\int_2^{\infty} \frac{dx}{x}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{e}{2^n}$

(e) $$4 + \cfrac{4}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \cfrac{6}{6 + \cfrac{7}{\ddots}}}}}}$$

(f) $\displaystyle\lim_{n\to\infty} \frac{2n}{\sqrt[n]{n!}}$

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{n!}$

(a) $2e$

(b) $2e$

(c) **Not** $2e$

(d) $2e$

(e) $2e$

(f) $2e$

(g) $2e$

Solution: For all of these, a good approach is to try computing the values for a few n, say $n = 2, 5, 10, 100$. For e, f, and maybe b, that's the only approach you would be expected to do.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n!} = 2\sum_{n=0}^{\infty} \frac{1}{n!} = 2e^1 = 2e$

(b) $\displaystyle\lim_{n\to\infty} \log\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right) = \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$. Here, we set $x = 1/n$ and used L'Hopitals rule in the final step. From this, $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and $\displaystyle\lim_{n\to\infty} 2\left(1+\frac{1}{n}\right)^n = 2e$.

(c) $\displaystyle\int_2^{\infty} \frac{dx}{x} = \log(x)\big|_2^{\infty} = \infty$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{e}{2^n} = e\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = e\frac{1}{1-\frac{1}{2}} = 2e$, from the geometric series.

(e) This requires the theory of continued fractions. See, for example `http://people.math.binghamton.edu/dikran/478/Ch7.pdf`, formula (7.14).

(f) Proving this formula requires Stirling's formula, $\log(n!) = n\log(n) - n + O(\log(n))$ where "big O" means that part grows no faster than $\log(n)$. Stirling's formula is proven by relating $\log(n!) = \sum_{x=1}^{n}\log(x)$ to the trapezoid rule approximation to the integral $\int_1^n \log(x)dx$. See `https://en.wikipedia.org/wiki/Stirling's_approximation` for details. With Stirling's approximation:

$$\log\frac{2n}{\sqrt[n]{n!}} = \log(2) + \log(n) - \frac{\log(n!)}{n} = \log(2) + \log(n) - \log(n) + 1 - O(\log(n))/n$$

so $\lim_{n\to\infty}\log\frac{2n}{\sqrt[n]{n!}} = \log(2) + 1$. Taking exp of both sides gives $\lim\limits_{n\to\infty}\dfrac{2n}{\sqrt[n]{n!}} = e^{\log(2)+1} = 2e$.

(g) $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{n!} = \sum_{n=0}^{\infty}\frac{n}{n!} + \sum_{n=0}^{\infty}\frac{1}{n!} = \left(\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\right) + e = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\right) + e = 2e.$

(10) 9. Find the sum of the geometric series $\frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \frac{5}{81} + \frac{5}{243} + \cdots$.

Solution:

$$\frac{5}{3}\cdot\frac{1}{1-\frac{1}{3}} = \frac{5}{2}$$

(10) 10. Define a sequence by $a_0 = 1$, $a_n = \frac{1+a_{n-1}}{2+a_{n-1}}$. Write out the first five terms of this sequence. Bonus: What can you say about the limit of this sequence?

Solution: $1, \frac{2}{3}, \frac{5}{8}, \frac{13}{21}, \frac{34}{55}, \ldots$ The sequence is decreasing and bounded below (by 0), so it has a limit. The limit satisfies $L = \frac{1+L}{2+L}$, so $L = \frac{\sqrt{5}-1}{2} = 0.618\ldots$

(10) 11. Match the description to the series:

(a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

(b) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

(c) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots$

(d) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

(e) Giants beat Rangers in 5 games.

1. Power series
2. Harmonic series
3. Geometric series
4. Alternating series
5. World series

Solution: (a) is geometric, (b) is harmonic, (c) is alternating, (d) is a power series, (e) was the 2010 World Series.

(10) 12. The Fourier series for $f(x) = x$ on the interval $[-\pi, \pi]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

where

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx.$$

Compute the coefficients b_n and write the first five terms of the Fourier series for f.

Solution: $b_n = \frac{-2\cos(n\pi)}{n}$, so

$$f(x) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \frac{2}{5}\sin(5x) + \cdots$$

(10) 13. Find the Taylor series for $f(x) = e^x$ at the point $x = 1$. Write using summation notation or show at least five terms.

Solution:

$$f(x) = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4 + \cdots$$

(10) 14. Find the fifth derivative of $f(x) = \dfrac{x}{1-x^2}$ at $x = 0$.

Solution:

$$f(x) = \frac{x}{1-x^2} = x(1 + x^2 + x^4 + x^6 + \cdots) = x + x^3 + x^5 + x^7 + \cdots$$

so $f^{(5)}(0) = 5! = 120$.

(10) 15. Give an example of a power series centered at 3 with radius of convergence equal to 5. Does your series converge at $x = 0$? Does it converge at $x = 8$?

Solution: One possible solution is:

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{5^n}$$

The ratio test gives $r = \left|\frac{x-3}{5}\right|$. The series converges when $r < 1$, so $-5 < x - 3 < 5$ or $-2 < x < 8$. When $x = 8$, the series becomes $\sum 1$ which diverges.

So, this series does converge at $x = 0$ but not at $x = 8$.