Math 1520 – Sample Exam 2

You may use a graphing calculator (TI-83, 84, for example) on this exam, but not one that can perform symbolic integration.

There are 10 questions, worth a total of 100 points.

(10) 1. Let A be the region bounded by $y = x^4$, the x-axis, and the line $x = 1$. Find the volume of the solid obtained by rotating A around the x-axis.

Solution:

$$\int_0^1 \pi(x^4)^2 dx = \frac{\pi}{9}x^9\Big|_0^1 = \frac{\pi}{9}.$$

(10) 2. Find the centroid of the region bounded by $y = \frac{1}{x}$ and the lines $y = \frac{1}{2}$ and $x = \frac{1}{2}$.

Solution: The area of the region is

$$\int_{1/2}^2 \left(\frac{1}{x} - \frac{1}{2}\right) dx = \log(4) - \frac{3}{4} \approx 0.636$$

The x moment is

$$\int_{1/2}^2 x\left(\frac{1}{x} - \frac{1}{2}\right) dx = \frac{9}{16} = 0.5625$$

So $\bar{x} = \frac{9/16}{\log(4)-3/4} = \frac{9}{16\log 4 - 12} \approx 0.884$. By symmetry $\bar{y} = \bar{x}$.

(10) 3. The functions $x(t) = a\cos(t)$ and $y(t) = b\sin(t)$ define an ellipse, for $0 \le t \le 2\pi$.

Set up an integral to find the circumference (arclength) of this ellipse.

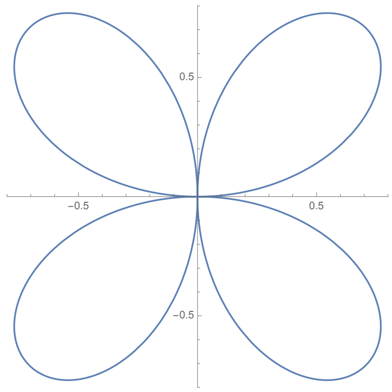
You do not need to work out the integral. It is called an *elliptic integral* and cannot be computed, except numerically.

Solution:

$$\int_0^{2\pi} \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}dt$$

(10) 4. Sketch an accurate plot of the curve given in polar coordinates by $r = \sin(2\theta)$ for $\theta \in [0, 2\pi]$. (Hint: it looks like a clover with four petals)

Solution:



(10) 5. Find the area of one petal of the curve in question 4. (You probably don't want to do this integral by hand, but this is a sample exam so it's ok.)

Solution: The area

$$A = \int_0^{\pi/2} \frac{1}{2} \sin^2(2\theta) d\theta = \frac{\pi}{8}.$$

(10) 6. Give an example of a sequence $\{a_n\}$ which has these three properties:

- All terms are positive ($a_n > 0$ for all n).
- The sequence is not bounded above.
- The sequence $\frac{1}{a_n}$ does not converge.

Solution: One possible example would be: $1, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, 7, \ldots$.

If you like your sequences to come with simple formulas, try $a_n = e^{(-1)^n n}$.

Another example is $a_n = \frac{1}{\sin^2(n)}$. Why is this unbounded? Hint: $\pi \approx 22/7$, so $22 \approx 7\pi$. What is a_{22}? A better approximation to π is $355/113$. What is a_{355}?

(10) 7. Let $a_n = e^{\left(\frac{n-1}{n+1}\right)}$ for $n \geq 1$. Does the sequence $\{a_n\}$ converge or diverge? If it converges, find its limit.

Solution: It converges to e.

(10) 8. Write the series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

using summation notation (use a $\sum$). Then compute its value.

Solution:

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1-(-1/2)} = 2/3.$$

(10) 9. For each series, decide if it converges or diverges (no explanation required).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{10n^2 - n - 3}{n^4 - n^3 - n^2 + 2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{1}{n}}$

(c) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

(d) $1000 + \frac{1000}{2} + \frac{1000}{4} + \frac{1000}{8} + \frac{1000}{16} + \frac{1000}{32} + \cdots$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n}{e^n}$

Solution:

(a) Converges. Just looking at the biggest powers, this is comparable to

$$\sum_{n=1}^{\infty} \frac{10n^2}{n^4} = 10 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is a p series with $p > 1$ and converges. However, to use the actual comparison test is a bit tricky here. The numerator is less than $10n^2$, so that's easy. For the denominator, if $n > 4$ then $n^4 > 4n^3$ so $n^3 < n^4/4$. Also $n^2 < n^3 < n^4/4$. So

$$n^4 - n^3 - n^2 + 2 > n^4 - n^4/4 - n^4/4 = n^4/2.$$

Then for $n > 4$,

$$\frac{10n^2 - n - 3}{n^4 - n^3 - n^2 + 2} < \frac{10n^2}{\frac{1}{2}n^4} = \frac{20}{n^2}$$

and we can use the comparison test.

(b) Diverges. This is a p series with $p = \frac{1}{2} < 1$.
Or use the integral test with $\int_1^\infty x^{-1/2} dx = \infty$.

(c) Diverges. This is the harmonic series.

(d) Converges (to 2000). This is a geometric series with common ratio 1/2.

(e) Diverges. This is a geometric series with common ratio $r = \pi/e$. Since $\pi > e$, the ratio $r > 1$. The terms get large, they don't go to zero.

(10) 10. Find the first five partial sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10^n}$.

Solution: 0.1, 0.12, 0.123, 0.1234, 0.12345