Math 370 – Sample Vector Calculus Questions

(10) 1. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and let $\varphi(x, y, z) = x + y + z$.

(a) Which one is defined, $\nabla\mathbf{F}$ or $\nabla\varphi$? Calculate it.

Solution: $\nabla\varphi = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

(b) Which one is defined, $\nabla \cdot \mathbf{F}$ or $\nabla \cdot \varphi$? Calculate it.

Solution: $\nabla \cdot \mathbf{F} = 3$.

(c) Which one is defined, $\nabla \times \mathbf{F}$ or $\nabla \times \varphi$? Calculate it.

Solution: $\nabla \times \mathbf{F} = 0$.

(10) 2. Find a unit normal vector to the surface $x^2 - xy + 4z = 11$ at the point $P = (3, 6, 5)$.

Solution: $\frac{1}{5}(-3\mathbf{j} + 4\mathbf{k})$ (or the negative of this vector).

(10) 3. Which of these vector fields are conservative?

(a) $\mathbf{F}(x, y) = y^2\mathbf{i} - xy\mathbf{j}$

(b) $\mathbf{F}(x, y) = \mathbf{i} - \mathbf{j}$

(c) $\mathbf{F}(x, y) = \frac{1}{x^2+y^2}(-y\mathbf{i} + x\mathbf{j})$

(d) $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$

(e) $\mathbf{F}(x, y) = e^x\mathbf{i} + e^y\mathbf{j}$

Solution: (b) and (e) are conservative.

(10) 4. Find a potential function for $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + \mathbf{k}$.

Solution: $\varphi(x, y, z) = xy + z$

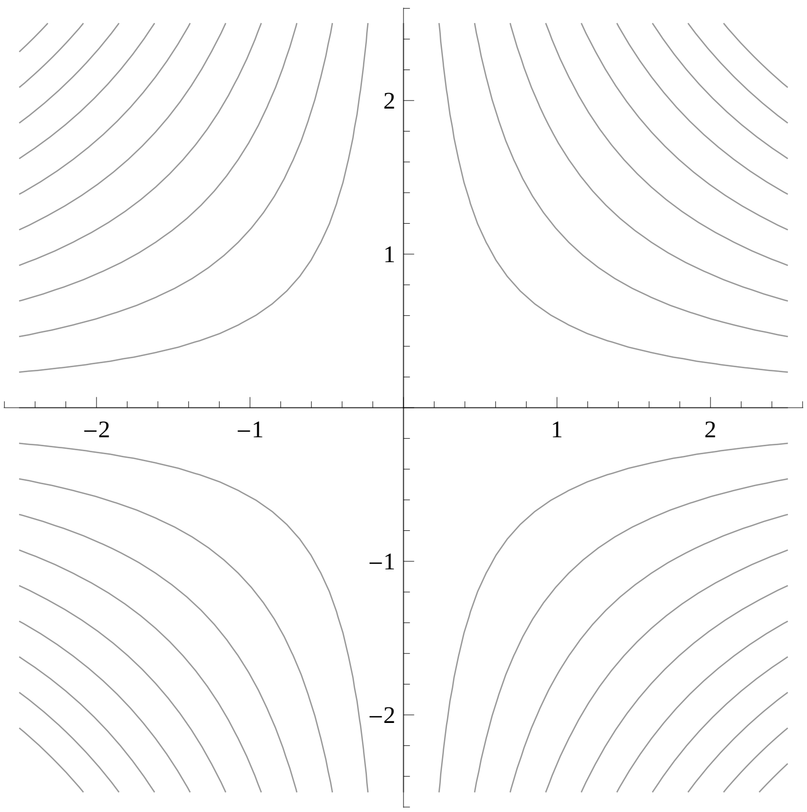
(10) 5. For $\mathbf{F}(x, y, z)$ a vector field, and $\varphi(x, y, z)$ a scalar function, prove the product rule:

$$\nabla \cdot \varphi\mathbf{F} = \nabla\varphi \cdot \mathbf{F} + \varphi\nabla \cdot \mathbf{F}$$

Solution: Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$. Then

$$\begin{aligned}
\nabla \cdot \varphi \mathbf{F} &= \frac{\partial}{\partial x}(\varphi f) + \frac{\partial}{\partial y}(\varphi g) + \frac{\partial}{\partial z}(\varphi h) && (1)\\
&= \frac{\partial \varphi}{\partial x} f + \varphi \frac{\partial f}{\partial x} + \frac{\partial \varphi}{\partial y} g + \varphi \frac{\partial g}{\partial y} + \frac{\partial \varphi}{\partial z} h + \varphi \frac{\partial h}{\partial z} && (2)\\
&= \frac{\partial \varphi}{\partial x} f + \frac{\partial \varphi}{\partial y} g + \frac{\partial \varphi}{\partial z} h + \varphi \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) && (3)\\
&= \nabla \varphi \cdot \mathbf{F} + \varphi \nabla \cdot \mathbf{F} && (4)
\end{aligned}$$

(10) 6. Level curves for $\varphi(x, y) = \frac{xy}{2}$ are shown below. On the same picture, accurately sketch the vector field $\nabla \varphi$. Plot at least four vectors in every quadrant, plus some on the axes.



(10) 7. Let C be any curve from (-1,4,0) to (3,0,7). Calculate $\int_C 2xdx + zdy + ydz$.

Solution: Potential function $\varphi(x, y, z) = x^2 + zy$, so $\int_C 2xdx + zdy + ydz = 9 - 1 = 8$.

(10) 8. Let $\mathbf{F}(x, y) = (x - 3y, x^2 + 4)$, and let C be the curve that goes once counterclockwise

around the rectangle with corners (0,0), (8,0), (8,3), and (0,3). Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

> **Solution:** By Green's Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \int_0^8 (2x + 3) dx dy = 264$.

(10) 9. For $u \geq 0, v \in [0, 2\pi]$, the parameterization $x = u \cos v, y = u \sin v, z = \frac{u}{3}$ describes an infinite cone. Find a unit normal vector to this cone as a function of u and v.

> **Solution:** The unit normal field is $\mathbf{n} = -\frac{1}{\sqrt{10}}(\cos v\mathbf{i} + \sin v\mathbf{j}) + \frac{3}{\sqrt{10}}\mathbf{k}$ (or the negative of this vector).

(10) 10. Find the flux of the vector field $F = e^{-z^2}\mathbf{k}$ through the cone from problem 9.

> **Solution:** 9π using the normal field $\mathbf{n}$ from problem 9, or -9π if you used $-\mathbf{n}$.