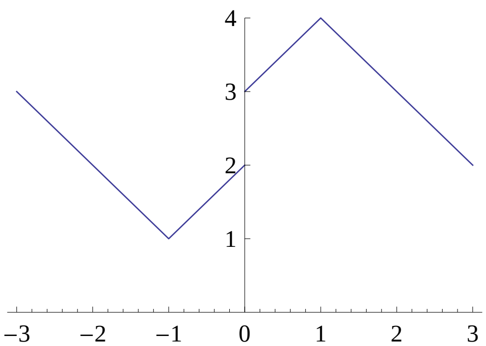
Math 370 – Sample Exam 2

(10) 1. Let $f(x)$ be the function on $[-3, 3]$ which is graphed below. Find the constant term in the Fourier series for f.

> **Solution:** The constant term is 5/2, the average value of f. The term $a_0 = 5$.



(10) 2. Let $f(x) = |x|$ for $-2 \leq x \leq 2$, and let $g(x) = 4|x| + 3$ for $-2 \leq x \leq 2$. The Fourier series for f is given by

$$f(x) = 1 - \frac{8}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right)$$

What is the Fourier series for g?

> **Solution:**
>
> $$g(x) = 4f(x) + 3 = 7 - \frac{32}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right)$$

(10) 3. Let $f(x) = \begin{cases} 0 & \text{for } -\pi \leq x < -\pi/2 \\ 1 & \text{for } -\pi/2 \leq x < \pi/2 \\ 0 & \text{for } \pi/2 < x \leq \pi \end{cases}$.

Find the Fourier series for f on the interval $[-\pi, \pi]$.

Give at least four terms in the series or write it as a summation.

> **Solution:**
>
> $$\frac{1}{2} + \frac{2\cos(x)}{\pi} - \frac{2\cos(3x)}{3\pi} + \frac{2\cos(5x)}{5\pi} - \cdots$$

(10) 4. The function f (shown below) is defined on the interval $[-2, 2]$. What value does the Fourier series for f converge to:

(a) When $x = -1$?

(a) **1/2**

(b) When $x = 0$?

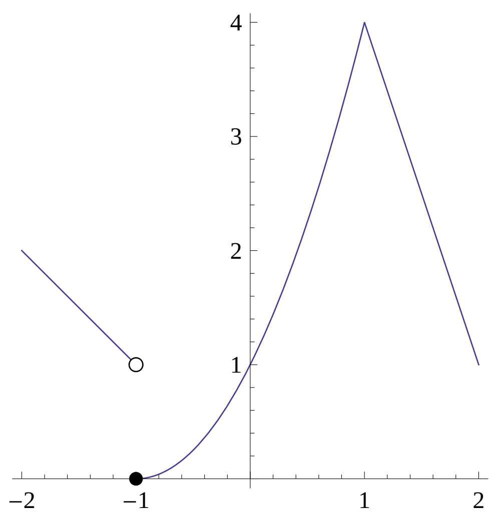
(b) **1**

(c) When $x = 1$?

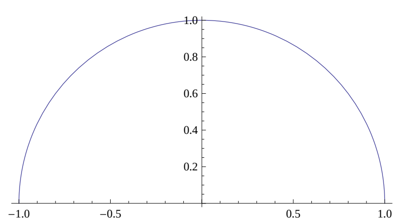
(c) **4**

(d) When $x = 2$?

(d) **3/2**



(10) 5. Give a clockwise parameterization of the semi-circle shown below:



Solution: Many solutions, one is $\mathbf{F}(t) = (-\cos(t), \sin(t))$ for $0 \leq t \leq \pi$. Another is $\mathbf{F}(t) = (t, \sqrt{1 - t^2})$ for $-1 \leq t \leq 1$.

(10) 6. Let $\mathbf{F}(t) = (2\cos(t), \sin(t))$. Sketch this curve for $0 \leq t \leq 2\pi$. Find the acceleration vector, and find four places where the tangential acceleration is zero.

Solution: The curve is an ellipse through $(2,0)$,$(0,1)$,$(-2,0)$, and $(0,-1)$, sketch that. $\mathbf{a}(t) = (-2\cos(t), -\sin(t))$, and the tangential acceleration is zero at the four points just listed.

(10) 7. Let $\mathbf{F}(t) = (t, \cosh(t))$, a catenary curve. Find the curvature of $\mathbf{F}$.
(Hint: The identity $\cosh^2(t) = 1 + \sinh^2(t)$ is helpful).

Solution: $\kappa(t) = \frac{1}{\cosh^2(t)}$

(10) 8. Let $\mathbf{F} = \mathbf{i} + 3y\mathbf{j} - 3z\mathbf{k}$. Find the streamline for $\mathbf{F}$ through the point (0,2,3).

Solution: With $\lambda\sigma' = \mathbf{F}$, we get $x' = \lambda$, $y' = 3y\lambda$, $z' = -3z\lambda$, or

$$dx = \frac{dy}{3y} = -\frac{dz}{3z}$$

Integrating pairs (both using dx) gives

$$x = \frac{1}{3}\ln|y| + c_1, x = -\frac{1}{3}\ln|z| + c_2$$

Solving gives

$$y = k_1e^{3x}, z = k_2e^{-3x}$$

for real numbers k_1, k_2. To go through (0,2,3), we get

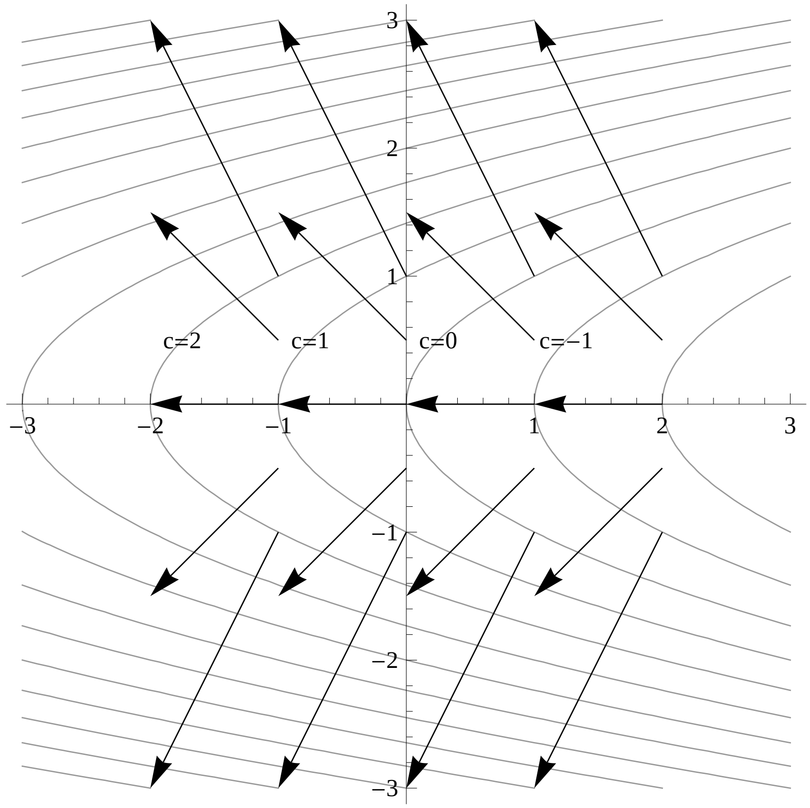
$$2 = k_1e^0, 3 = k_2e^0$$

so the streamline, parameterized by x, is given by

$$\sigma(x) = x\mathbf{i} + 2e^{3x}\mathbf{j} + 3e^{-3x}\mathbf{k}.$$

Note that integrating the y and z expressions gives $y = 6/z$ which can lead to other correct answers with different parameterizations.

(10) 9. Let $f(x,y) = y^2 - x$. Accurately draw level curves for f and the vector field ∇f on the same sketch.



Solution:

(10) 10. Let $\mathbf{P}(t)$ be a parameterized curve with velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$. Suppose $m > 0$ is constant. If the curve satisfies $\mathbf{F} = m\mathbf{a}$ for some vector field $\mathbf{F}$ along the curve, prove

$$\frac{d}{dt}(m\mathbf{P} \times \mathbf{v}) = \mathbf{P} \times \mathbf{F}.$$

(The physical meaning of this is that rate of change of angular momentum is the torque.)

Solution:

$$\begin{aligned}\frac{d}{dt}(m\mathbf{P} \times \mathbf{v}) &= \frac{d}{dt}(m\mathbf{P}) \times \mathbf{v} + m\mathbf{P} \times \frac{d\mathbf{v}}{dt} \\ &= m\mathbf{v} \times \mathbf{v} + \mathbf{P} \times m\mathbf{a} \\ &= m\mathbf{0} + \mathbf{P} \times \mathbf{F} \\ &= \mathbf{P} \times \mathbf{F}\end{aligned}$$