Math 370 – Sample Exam 1

You may use a calculator on this exam. There are 10 questions, worth a total of 100 points.

(10) 1. Find the angle between the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j} + \sqrt{3}\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} - \sqrt{3}\mathbf{k}$.

(10) 2. Find all solutions to the system of equations shown below:

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 - x_5 &= 0 \\ x_2 + x_4 &= 0 \\ x_3 - x_5 &= 0 \end{aligned}$$

(10) 3. Let $\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -3 \\ -1 & 0 & 1 \end{pmatrix}$.

(a) Find $\mathbf{M}_R$, the row reduced echelon matrix equivalent to $\mathbf{M}$ (show your work).

(b) What is rank $\mathbf{M}$?

(c) What is the nullity of $\mathbf{M}$?

(d) What does the rank-nullity theorem say for this matrix?

(10) 4. Suppose $\mathbf{A}$ is a 2x3 matrix, $\mathbf{B}$ is 2x4, and $\mathbf{C}$ is 4x3. What size is:

(a) $\mathbf{BC}$

(b) $\mathbf{AC}^t$

(c) $\mathbf{CA}^t\mathbf{B}$

(10) 5. A matrix $\mathbf{A}$ is *symmetric* if $\mathbf{A} = \mathbf{A}^t$. Give an example of a 3x3 symmetric matrix which contains all of the numbers 1,2,3,4,5, and 6 as entries.

(10) 6. A square matrix is *orthogonal* if its transpose is equal to its inverse.
Suppose $\mathbf{A}$ is symmetric, $\mathbf{B}$ is orthogonal, and both matrices are the same size.

Show that $\mathbf{BAB}^{-1}$ is symmetric.

(10) 7. Let $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(a) Explain geometrically what multiplication by $\mathbf{M}$ does to vectors in the plane.

(b) Find $\mathbf{M}^{-1}$ and explain what multiplication by $\mathbf{M}^{-1}$ does to vectors in the plane.

(10) 8. Subspace or not a subspace? Answer for each part.

(a) S is all vectors (x, y, z) in $\mathbb{R}^3$ with $x + y + z = 0$.

(b) S is all vectors in $\mathbb{R}^4$ of the form $(x, y, x + 1, y + 1)$.

(c) S is the line $y = x$ in $\mathbb{R}^2$.

(d) S is the parabola $y = x^2$ in $\mathbb{R}^2$.

(e) S consists of the two lines $y = x$ and $y = -x$ in $\mathbb{R}^2$.

(10) 9. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & -\frac{1}{2} & 8 & \frac{1}{6} \\ 28 & -18 & 12 & 12 & 24 \end{pmatrix}$.

The matrices below are each formed from $\mathbf{A}$ by a single row operation.

Given that $\det \mathbf{A} = 99$, compute:

(a) $\det \begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & -\frac{1}{2} & 8 & \frac{1}{6} \\ 28 & -18 & 12 & 12 & 24 \end{pmatrix} =$

(b) $\det \begin{pmatrix} 28 & -18 & 12 & 12 & 24 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & -\frac{1}{2} & 8 & \frac{1}{6} \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} =$

(c) $\det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 55 & 1 & 55 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & -\frac{1}{2} & 8 & \frac{1}{6} \\ 28 & -18 & 12 & 12 & 24 \end{pmatrix} =$

(10) 10. Let $\mathbf{A}$ be an $n \times n$ square matrix with 1's on the diagonal and -1's below the diagonal, as shown here:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots\dots\dots & & 0 \\ -1 & 1 & 0 & 0 & \dots\dots\dots & & 0 \\ 0 & -1 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1 & 0 & 0 \\ 0 & \dots\dots & & 0 & -1 & 1 & 0 \\ 0 & \dots\dots & & 0 & 0 & -1 & 1 \end{pmatrix}$$

Compute the matrix inverse $\mathbf{A}^{-1}$.