Math 370 – Sample Exam 1

You may use a calculator on this exam. There are 10 questions, worth a total of 100 points.

(10) 1. Find the angle between the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j} + \sqrt{3}\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} - \sqrt{3}\mathbf{k}$.

> **Solution:** $\mathbf{u} \cdot \mathbf{v} = -1$, $||u|| = ||v|| = \sqrt{5}$, so $\cos(\theta) = \frac{-1}{\sqrt{5}\sqrt{5}}$, so $\theta \approx 1.77$ (or $\theta \approx 101.5°$).

(10) 2. Find all solutions to the system of equations shown below:

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 - x_5 &= 0 \\ x_2 + x_4 &= 0 \\ x_3 - x_5 &= 0 \end{aligned}$$

> **Solution:** All solutions are of the form $(0, -x_4, x_5, x_4, x_5)$ with x_4 and x_5 free.

(10) 3. Let $\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -3 \\ -1 & 0 & 1 \end{pmatrix}$.

(a) Find $\mathbf{M}_R$, the row reduced echelon matrix equivalent to $\mathbf{M}$ (show your work).

> **Solution:** $\mathbf{M}_R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) What is rank $\mathbf{M}$?

(c) What is the nullity of $\mathbf{M}$?

(d) What does the rank-nullity theorem say for this matrix?

> **Solution:** The rank is 2, nullity is 1, and the rank-nullity theorem says that $2 + 1 = 3$ is the width of the matrix.

(10) 4. Suppose $\mathbf{A}$ is a 2x3 matrix, $\mathbf{B}$ is 2x4, and $\mathbf{C}$ is 4x3. What size is:

(a) $\mathbf{BC}$

(b) $\mathbf{AC}^t$

(c) $\mathbf{CA}^t\mathbf{B}$

Solution: $\mathbf{BC}$ is 2x3, $\mathbf{AC}^t$ is 2x4, $\mathbf{CA}^t\mathbf{B}$ is 4x4.

(10) 5. A matrix $\mathbf{A}$ is *symmetric* if $\mathbf{A} = \mathbf{A}^t$. Give an example of a 3x3 symmetric matrix which contains all of the numbers 1,2,3,4,5, and 6 as entries.

Solution: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

(10) 6. A square matrix is *orthogonal* if its transpose is equal to its inverse.
Suppose $\mathbf{A}$ is symmetric, $\mathbf{B}$ is orthogonal, and both matrices are the same size.
Show that $\mathbf{BAB}^{-1}$ is symmetric.

Solution: $(\mathbf{BAB}^{-1})^t = (\mathbf{B}^{-1})^t\mathbf{A}^t\mathbf{B}^t = (\mathbf{B}^t)^t\mathbf{AB}^{-1} = \mathbf{BAB}^{-1}$, so $\mathbf{BAB}^{-1}$ is symmetric.

(10) 7. Let $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(a) Explain geometrically what multiplication by $\mathbf{M}$ does to vectors in the plane.

(b) Find $\mathbf{M}^{-1}$ and explain what multiplication by $\mathbf{M}^{-1}$ does to vectors in the plane.

Solution: a. Multiplication by $\mathbf{M}$ rotates vectors 90° clockwise.
b. $\mathbf{M}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which rotates vectors 90° counterclockwise.

(10) 8. Subspace or not a subspace? Answer for each part.

(a) S is all vectors (x, y, z) in $\mathbb{R}^3$ with $x + y + z = 0$.

(b) S is all vectors in $\mathbb{R}^4$ of the form $(x, y, x + 1, y + 1)$.

(c) S is the line $y = x$ in $\mathbb{R}^2$.

(d) S is the parabola $y = x^2$ in $\mathbb{R}^2$.

(e) S consists of the two lines $y = x$ and $y = -x$ in $\mathbb{R}^2$.

Solution: a: Subspace; b: Not subspace; c: Subspace; d: Not subspace; e: Not subspace

(10) 9. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & -\frac{1}{2} & 8 & \frac{1}{6} \\ 28 & -18 & 12 & 12 & 24 \end{pmatrix}$.

The matrices below are each formed from $\mathbf{A}$ by a single row operation.

Given that $\det \mathbf{A} = 99$, compute:

(a) $\det \begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & -\frac{1}{2} & 8 & \frac{1}{6} \\ 28 & -18 & 12 & 12 & 24 \end{pmatrix} =$

(b) $\det \begin{pmatrix} 28 & -18 & 12 & 12 & 24 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & -\frac{1}{2} & 8 & \frac{1}{6} \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} =$

(c) $\det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 55 & 1 & 55 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & -\frac{1}{2} & 8 & \frac{1}{6} \\ 28 & -18 & 12 & 12 & 24 \end{pmatrix} =$

Solution: a: 11; b: -99; c: 99

(10) 10. Let $\mathbf{A}$ be an $n \times n$ square matrix with 1's on the diagonal and -1's below the diagonal, as shown here:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \multicolumn{2}{c}{\cdots\cdots} & 0 \\ -1 & 1 & 0 & 0 & \multicolumn{2}{c}{\cdots\cdots} & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\cdots\cdots} & 0 & -1 & 1 & 0 \\ 0 & \multicolumn{2}{c}{\cdots\cdots} & 0 & 0 & -1 & 1 \end{pmatrix}$$

Compute the matrix inverse $\mathbf{A}^{-1}$.

Solution: $\mathbf{A}^{-1}$ is a matrix with 1's on the diagonal and in every entry below the diagonal:

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \multicolumn{2}{c}{\dots\dots\dots} & 1 & 1 & 0 \\ 1 & \multicolumn{2}{c}{\dots\dots\dots} & 1 & 1 & 1 \end{pmatrix}$$