Exercises

Chapter 8.2 # 1, 3, 9, 10

Chapter 8.3 # 11

Problem A: Let $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ be a rotation matrix. Compute the determinant of $\mathbf{R}_\theta$.

Problem B: If $\mathbf{A}$ is nonsingular, show that $\det(\mathbf{A}\mathbf{A}^t) > 0$.

Problem C: A matrix $\mathbf{A}$ is called *orthogonal* if $\mathbf{A}^t = \mathbf{A}^{-1}$. If $\mathbf{A}$ is orthogonal, what are the possible values for $\det \mathbf{A}$?

Problem D: A square matrix $\mathbf{A}$ is called *idempotent* if $\mathbf{A}^2 = \mathbf{A}$. If $\mathbf{A}$ is idempotent, what are the possible values for $\det \mathbf{A}$? Give examples of idempotent matrices with all possible values of the determinant.

Problem E: Pascal's triangle is a triangle of numbers in rows, with each number equal to the sum of the two above it:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

There are three natural ways to make a matrix out of Pascal's triangle. Here are examples with size 5:

$$
L_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \quad
U_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad
S_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}
$$

Check that $S_5 = L_5 U_5$. Use this to calculate $\det S_5$.

Problem F: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $M = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix}$.

Find $\det M$ and relate it to $\det A$ and $\det B$.